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"ELECTROSTATICS

AND

MAGNETISM"

BY

SIR WILLIAM THOMSON, D.C.L., LL.D., F.R.S., F.R.S.E.,

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PREFACE.

THIS volume consists chiefly of reprinted articles, on Electrostatics and mathematically allied subjects, which originally appeared at different times during the last thirty years, in the *Cambridge Mathematical Journal*, the *Cambridge and Dublin Mathematical Journal*, Liouville's *Journal de Mathématiques*, the *Philosophical Magazine*, Nichol's *Cyclopædia*, the *Reports of the British Association*, the *Transactions* or *Proceedings of the Royal Societies of London and Edinburgh*, the *Royal Institution of Great Britain*, and the *Philosophical Societies of Manchester and Glasgow*. The remainder, constituting about a quarter of the whole, is now printed for the first time from manuscript, which, except a small part about twenty years old, entitled "Electromagnets," has been written for the present publication, to fill up roughly gaps in the collection. The original dates of the republished articles, the dates of all new matter appearing as insertions or notes in the course of those articles, and the dates of the fresh articles have all been carefully indicated.

The article on Atmospheric Electricity, extracted from Nichol's *Cyclopædia*, was originally written at the request of my late friend and colleague the Editor; and for the permission to reprint it I am indebted to his son, my colleague, Professor John Nichol, and to the Messrs. Griffin, the publishers of the *Cyclopædia*.

The present volume includes as nearly as may be all that I have hitherto written on electrostatics and magnetism. I have excluded from it electrical papers in which either thermodynamics or the kinetics of electricity is prominent. I intend that, as soon as possible, it shall be followed by a collected reprint of all my other papers hitherto published.

I take this opportunity of thanking Professors Clerk Maxwell and Tait for much valuable assistance which they have given me in the course of this work.

WILLIAM THOMSON.

YACHT "LALLA ROOKH,"
LAMLASH, *Oct.* 12, 1872.

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I.—ON THE UNIFORM MOTION OF HEAT IN HOMOGENEOUS SOLID BODIES, AND ITS CONNEXION WITH THE MATHEMATICAL THEORY OF ELECTRICITY.*

[From *Cambridge Mathematical Journal*, Feb. 1842. Reprinted *Philosophical Magazine* (1854, first half-year).]

[Since the following article was written,† the writer finds that most of his ideas have been anticipated by M. Chasles in two Mémoires in the *Journal de Mathématiques;* the first, in vol. III., on the Determination of the Value of a certain Definite Integral, and the second, in vol. V., on a new Method of Determining the Attraction of an Ellipsoid on a Point without it. In the latter of these Mémoires, M. Chasles refers to a paper, by himself, in the twenty-fifth *Cahier* of the *Journal de l'Ecole Polytechnique,* in which it is probable there are still further anticipations, though the writer of the present article has not had access to so late a volume of the latter journal. Since, however, most of his methods are very different from those of M. Chasles, which are nearly entirely geometrical, the following article may be not uninteresting to some readers:—]

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* [*Note added June* 1854].—This paper first appeared anonymously in the *Cambridge Mathematical Journal* in February 1842. The text is reprinted without alteration or addition. All the footnotes are of the present date (March 1854). The general conclusions established in it show that the laws distribution of electric or magnetic force in any case whatever must be identical with the laws of distribution of the lines of motion of heat in certain perfectly defined circumstances. With developments and applications contained in a subsequent paper (II. below) on the Elementary Laws of Statical Electricity (*Cambridge and Dublin Mathematical Journal,* Nov. 1845), they constitute a full theory of the characteristics of lines of force, which have been so admirably investigated experimentally by Faraday, and complete the analogy with the theory of the conduction of heat, of which such terms as "conducting power for lines of force" (*Exp. Res.* §§ 2797-2802) involve the idea.

† [*Note added June* 1854].—This preliminary notice was written some months later than the text which follows, and was communicated to the editor of the journal to be prefixed to the paper, which had been in his hands since the month of September 1841. The ideas in which the author had ascertained he had been anticipated by M. Chasles, were those by which he was led to the determination of the attraction of an ellipsoid given in the latter part of the paper. He found soon afterwards that he was anticipated by the same author in an enunciation of the general theorems regarding attraction; still later he found that both an enunciation and demonstration of the same general theorems had been given by Gauss, whose paper ap-

1. If an infinite homogeneous solid be submitted to the action of certain constant sources of heat, the stationary temperature at any point will vary according to its position; and through every point there will be a surface, over the whole extent of which the temperature is constant, which is therefore called an *isothermal* surface. In this paper the case will be considered in which these surfaces are finite, and consequently closed.

2. It is obvious that the temperature of any point without a given isothermal surface, depends merely on the form and temperature of the surface, being independent of the actual sources of heat by which this temperature is produced, provided there are no sources without the surface. The temperature of an external point is consequently the same as if all the sources were distributed over this surface in such a manner as to produce the given constant temperature. Hence we may consider the temperature of any point without the isothermal surface, as the sum of the temperatures due to certain constant sources of heat, distributed over that surface.

---

peared shortly after M. Chasles' enunciations; and after all, he found that these theorems had been discovered and published in the most complete and general manner, with rich applications to the theories of electricity and magnetism, more than ten years previously, by Green! It was not until early in 1845 that the author, after having inquired for it in vain for several years, in consequence of an obscure allusion to it in one of Murphy's papers, was fortunate enough to meet with a copy of the remarkable paper ("An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," by George Green, Nottingham, 1828) in which this great advance in physical mathematics was first made. It is worth remarking, that, referring to Green as the originator of the term, Murphy gives a mistaken definition of "potential." It appears highly probable that he may never have had access to Green's essay at all, and that this is the explanation of the fact (of which any other explanation is scarcely conceivable), that in his Treatise on Electricity (Murphy's *Electricity*, Cambridge, 1833) he makes no allusion whatever to Green's discoveries, and gives a theory in no respect pushed beyond what had been done by Poisson. All the general theorems on attraction which Green and the other writers referred to, demonstrated by various purely mathematical processes, are seen as axiomatic truths in approaching the subject by the way laid down in the paper which is now republished. The analogy with the conduction of heat on which these views are founded, has not, so far as the author is aware, been noticed by any other writer.

3. To find the temperature produced by a single source of heat, let r be the distance of any point from it, and let v be the temperature at that point. Then, since the temperature is the same for all points situated at the same distance from the source, it is readily shown that v is determined by the equation

$$-r^2\frac{dv}{dr}=A.$$

$-\frac{A}{r}$

Dividing both members by r^2, and integrating, we have

$$v=\frac{A}{r}+C.$$

Now let us suppose that the natural temperature of the solid, or the temperature at an infinite distance from the source, is zero: then we shall have $C=0$, and consequently

$$v=\frac{A}{r} \qquad (1).$$

4. Hence that part of the temperature of a point without an isothermal surface which is due to the sources of heat situated on any element, $d\omega_1^2$, of the surface, is $\frac{\rho_1 d\omega_1^2}{r_1}$, where r_1 is the distance from the element to that point, and ρ_1 a quantity measuring the intensity of the sources of heat at different parts of the surface. Hence, the supposition being still made that there are no sources of heat without the surface if v be the temperature at the external point, we have

$$v=\iint\frac{\rho_1 d\omega_1^2}{r_1} \qquad (2),$$

the integrals being extended over the whole surface. The quantity ρ_1 must be determined by the condition

$$v=v_1 \qquad (3),$$

for any point in the surface, v_1 being a given constant temperature.

5. Let us now consider what will be the temperature of a point within the surface, supposing all the sources of heat by which the surface is retained at the temperature v_1 to be distributed over it. Since there are no sources in the interior of the surface, it follows that as much heat must flow out from the interior across the surface as flows into the interior, from the sources of heat at the surface. Hence the total flux of heat from the original surface to an adjacent isothermal surface in

the interior is nothing. Hence also the flux of heat from this latter surface to an adjacent isothermal surface in its interior must be nothing; and so on through the whole of the body within the original surface. Hence the temperature in the interior is constant, and equal to v_1, and therefore, for points at the surface, or within it, we have

$$\iint \frac{\rho_1 d\varpi_1{}^2}{r_1} = v_1 \qquad (4).$$

Now, if we suppose the surface to be covered with an attractive medium, whose density at different points is proportional to ρ_1, $-\frac{d}{dx}\iint \frac{\rho_1 d\varpi_1{}^2}{r_1}$ will be the attraction, in the direction of the axis of x, on a point whose rectangular co-ordinates are x, y, z. Hence it follows that the attraction of this medium on a point within the surface is nothing, and consequently ρ_1 is proportional to the intensity of electricity in a state of equilibrium on the surface, the attraction of electricity in a state of equilibrium being nothing on an interior point. Since, at the surface, the value of $\iint \frac{\rho_1 d\varpi_1{}^2}{r_1}$ is constant, and since, on that account, its value within the surface is constant also, it follows, that if the attractive force on a point at the surface is perpendicular to the surface, the attraction on a point within the surface is nothing. Hence the sole condition of equilibrium of electricity, distributed over the surface of a body, is, that it must be so distributed that the attraction on a point at the surface, oppositely electrified, may be perpendicular to the surface.

6. Since, at any of the isothermal surfaces, v is constant, it follows that $-\frac{dv}{dn}$, where n is the length of a curve which cuts all the surfaces perpendicularly, measured from a fixed point to the point attracted, is the total attraction on the latter point; and that this attraction is in a tangent to the curve n, or in a normal to the isothermal surface passing through the point. For the same reason also, if ρ_1 represent a flux of heat, and not an electrical intensity, $-\frac{dv}{dn}$ will be the total flux of heat at the variable extremity of n, and the direction of this flux will be

along n, or perpendicular to the isothermal surface. Hence, if a surface in an infinite solid be retained at a constant temperature, and if a conducting body, bounded by a similar surface, be electrified, the flux of heat, at any point, in the first case, will be proportional to the attraction on an electrical point, similarly situated, in the second; and the direction of the flux will correspond to that of the attraction.

7. Let $-\frac{dv_1}{dn_1}$ be the external value of $-\frac{dv}{dn}$ at the original surface, or the attraction on a point without it, and indefinitely near it. Now this attraction is composed of two parts; one the attraction of the adjacent element of the surface; and the other the attraction of all the rest of the surface. Hence, calling the former of these a, and the latter b, we have

$$-\frac{dv_1}{dn_1}=a+b.$$

Now, since the adjacent element of the surface may be taken as infinitely larger, in its linear dimensions, than the distance from it of the point attracted, its attraction will be the same as that of an infinite plane, of the density ρ_1. Hence a is independent of the distance of the point from the surface, and is equal to $2\pi\rho_1$. Hence

$$-\frac{dv_1}{dn_1}=2\pi\rho_1+b.$$

Now, for a point within the surface, the attraction of the adjacent element will be the same, but in a contrary direction, and the attraction of the rest of the surface will be the same, and in the same direction. Hence the attraction on a point within the surface, and indefinitely near it, is $-2\pi\rho_1+b$; and consequently, since this is equal to nothing, we must have $b=2\pi\rho_1$, and therefore

$$-\frac{dv_1}{dn_1}=4\pi\rho_1 \qquad (5).$$

Hence ρ_1 is equal to the total flux of heat, at any point of the surface, divided by 4π.

8. It also follows that if the attraction of matter spread over the surface be nothing on an interior point, the attraction on an exterior point, indefinitely near the surface, is perpendicular to the surface, and equal to the density of the matter at the part of the surface adjacent to that point, multiplied by 4π.

9. If v be the temperature at any isothermal surface, and ρ the intensity of the sources at any point of this surface, which would be necessary to sustain the temperature v, we have, by (5),

$$-\frac{dv}{dn}=4\pi\rho,$$

which equation holds, whatever be the manner in which the actual sources of heat are arranged, whether over an isothermal surface or not; and the temperature produced in an external point by the former sources, is the same as that produced by the latter. Also, the total flux of heat across the isothermal surface, whose temperature is v, is equal to the total flux of heat from the actual sources. From this, and from what has been proved above, it follows that if a surface be described round a conducting or non-conducting electrified body, so that the attraction on points situated on this surface may be everywhere perpendicular to it, and if the electricity be removed from the original body, and distributed in equilibrium over this surface, its intensity at any point will be equal to the attraction of the original body on that point, divided by 4π, and its attraction on any point without it will be equal to the attraction of the original body on the same point.*

If we call E the total expenditure of heat, or the whole flux across any isothermal surface, we have, obviously,

$$E=-\iint\frac{dv_1}{dn_1}d\omega_1{}^2.$$

10. Now this quantity should be equal to the sum of the expenditures of heat from all the sources. To verify this, we must, in the first place, find the expenditure of a single source. Now the temperature produced by a single source is, by (1), $v=\frac{A}{r}$, and hence the expenditure is obviously equal to

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* [*Note added June* 1854].—After having established this remarkable theorem in the manner shown in the text, the author attempted to prove it by direct integration, but only succeeded in doing so upwards of a year later, when he obtained the demonstration published in a paper, "Propositions in the Theory of Attraction" (*Camb. Math. Jour.* Nov. 1842), which appeared almost contemporaneously with a paper by M. Sturm in Liouville's *Journal*, containing the same demonstration; exactly the same demonstration, as the author afterwards (in 1845) found, had been given fourteen years earlier by Green.]

$-\frac{dv}{dr}\times 4\pi r^2$, or to $4\pi A$. If $A=\rho_1 d\omega_1^2$, this becomes $4\pi\rho_1 d\omega_1^2$. Hence the total expenditure is $\iint 4\pi\rho_1 d\omega_1^2$, or $-\iint\frac{dv_1}{dn_1}d\omega_1^2$, which agrees with the expression found above.

The following is an example of the application of these principles:—

Uniform Motion of Heat in an Ellipsoid.

11. The principles established above afford an easy method of determining the isothermal surfaces, and the corresponding temperatures, in the case in which the original isothermal surface is an ellipsoid.

The first step is to find ρ_1, which is proportional to the quantity of matter at any point in the surface of an ellipsoid, when the matter is so distributed that the attraction on a point within the ellipsoid is nothing. Now the attraction of a shell, bounded by two concentric similar ellipsoids, on a point within it, is nothing. If the shell be infinitely thin, its attraction will be the same as that of matter distributed over the surface of one of the ellipsoids in such a manner that the quantity on a given infinitely small area at any point is proportional to the thickness of the shell at the same point. Let a_1, b_1, c_1 be the semi-axes of one of the ellipsoids, $a_1+\delta a_1$, $b_1+\delta b_1$, $c_1+\delta c_1$ those of the other. Let also p_1 be the perpendicular from the centre to the tangent plane at any point on the first ellipsoid, and $p_1+\delta p_1$ the perpendicular from the centre to the tangent plane at a point similarly situated on the second. Then δp_1 is the thickness of the shell, since, the two ellipsoids being similar, the tangent planes at the points similarly situated on their surfaces are parallel. Also, on account of their similarity, $\frac{\delta a_1}{a_1}=\frac{\delta b_1}{b_1}=\frac{\delta c_1}{c_1}=\frac{\delta p_1}{p_1}$, and consequently the thickness of the shell is proportional to p_1. Hence we have, by (5),

$$-\frac{1}{4\pi}\frac{dv_1}{dn_1}=\rho_1=k_1 p_1 \qquad (a),$$

where k_1 is a constant, to be determined by the condition $v=v_1$, at the surface of the ellipsoid.

12. To find the equation of the isothermal surface at which the temperature is v_1+dv_1, let $-dv_1=C$, in (a). Then we have

$k_1p_1dn_1=\frac{C}{4\pi}$, or $p_1dn_1=\theta_1$, where θ_1 is an infinitely small constant quantity; and the required equation will be the equation of the surface traced by the extremity of the line dn_1, drawn externally perpendicular to the ellipsoid. Let x', y', z' be the co-ordinates of any point in that surface, and x, y, z those of the corresponding point in the ellipsoid. Then, calling $\alpha_1, \beta_1, \gamma_1$ the angles which a normal to the ellipsoid at the point whose co-ordinates are x, y, z makes with these co-ordinates, and supposing the axes of x, y, z to coincide with the axes of the ellipsoid, $2a_1$, $2b_1$, $2c_1$, respectively, we have

$$x'-x=dn_1\cos\alpha_1=\frac{\frac{x}{a_1^2}dn_1}{\sqrt{\left(\frac{x^2}{a_1^4}+\frac{y^2}{b_1^4}+\frac{z^2}{c_1^4}\right)}}=\frac{x}{a_1^2}p_1dn_1=\frac{x}{a_1^2}\theta_1.$$

or $x'-x=\frac{x'}{a_1^2}\theta_1$, since θ_1 is infinitely small, and therefore also $x'-x$; whence

$$x=x'\left(1-\frac{\theta_1}{a_1^2}\right)=\frac{x'}{1+\frac{\theta_1}{a_1^2}}.$$

In a similar manner we should find

$$y=\frac{y'}{1+\frac{\theta_1}{b_1^2}},\quad\text{and}\quad z=\frac{z'}{1+\frac{\theta_1}{c_1^2}}.$$

But $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}+\frac{z^2}{c_1^2}=1$, and hence we have

$$\frac{x'^2}{a_1^2\left(1+\frac{\theta_1}{a_1^2}\right)^2}+\frac{y'^2}{b_1^2\left(1+\frac{\theta_1}{b_1^2}\right)^2}+\frac{z'^2}{c_1^2\left(1+\frac{\theta_1}{c_1^2}\right)^2}=1,$$

or
$$\frac{x'^2}{a_1^2+2\theta_1}+\frac{y'^2}{b_1^2+2\theta_1}+\frac{z'^2}{c_1^2+2\theta_1}=1,$$

for the equation to the isothermal surface whose temperature is v_1+dv_1, and which is therefore an ellipsoid described from the same foci as the original isothermal ellipsoid. In exactly the same manner it might be shown that the isothermal surface whose temperature is $v_1+dv_1+dv_1'$, is an ellipsoid having the same foci as the ellipsoid whose temperature is v_1+dv_1, and consequently, as the original ellipsoid also. By continuing this process it may be proved that all the isothermal surfaces are ellipsoids, having the same foci as the original one.

13. From the form of the equation found above for the isothermal ellipsoid whose temperature is $v_1 + dv_1$, it follows that θ_1 or $p_1 dn_1$ is $= a_1 da_1$, where da_1 is the increment of a_1, corresponding to the increment dn_1 of n_1. Hence, if a be one of the semi-axes of an ellipsoid, $a + da$ the corresponding semi-axis of another ellipsoid having the same foci, dn the thickness at any point of the shell bounded by the two ellipsoids, and p the perpendicular from the centre to the plane touching either ellipsoid at the same point, we have

$$\frac{dn}{da} = \frac{a}{p} \qquad (b).$$

14. All that remains to be done is to find the temperature at the surface of any given ellipsoid, having the same foci as the original ellipsoid. For this purpose, let us first find the value of $-\frac{dv}{dn}$ at any point in the surface of the isothermal ellipsoid whose semi-axes are a, b, c. Now we have, from (a),

$$-\frac{dv}{dn} = 4\pi kp,$$

where k is constant for any point in the surface of the isothermal ellipsoid under consideration, and determined by the condition that the whole flux of heat across this surface must be equal to the whole flux across the surface of the original ellipsoid. Now the first of these quantities is equal to $4\pi k \iint p d\omega^2$ ($d\omega^2$ being an element of the surface), or to $4\pi \frac{ka}{\delta a} \iint \delta p d\omega^2$, since $\frac{\delta a}{a} = \frac{\delta p}{p}$. But $\iint \delta p d\omega^2$ is equal to the volume of a shell bounded by two similar ellipsoids, whose semi-axes are a, b, c, and $a + \delta a$, $b + \delta b$, $c + \delta c$, and is therefore readily shown to be equal to $4\pi \frac{\delta a}{a} abc$. Hence $4\pi \frac{ka}{\delta a} \iint \delta p d\omega^2$, or $4\pi k \iint p d\omega^2$, is equal to $4^2 \pi^2 kabc$. In a similar manner we have, for the flux of heat across the original isothermal surface, $4^2 \pi^2 k_1 a_1 b_1 c_1$, and therefore

$$4^2 \pi^2 kabc = 4^2 \pi^2 k_1 a_1 b_1 c_1,$$

which gives

$$k = k_1 \frac{a_1 b_1 c_1}{abc}.$$

Hence we have

$$-\frac{dv}{dn}=4\pi k_1\frac{a_1b_1c_1}{abc}p \qquad (c).$$

15. The value of v may be found by integrating this equation. To effect this, since a, b, c are the semi-axes of an ellipsoid passing through the variable extremity of n, and having the same foci as the original ellipsoid, whose axes are a_1, b_1, c_1, we have

$$a^2-a_1^{\,2}=b^2-b_1^{\,2}=c^2-c_1^{\,2};$$

which gives

$$\left.\begin{aligned} b^2&=a^2-f^2\\ c^2&=a^2-g^2\end{aligned}\right\} \qquad (d).$$

where $f^2=a_1^{\,2}-b_1^{\,2}$, $g^2=a_1^{\,2}-c_1^{\,2}$

Hence (c) becomes

$$-\frac{dv}{dn}=4\pi k_1\frac{a_1b_1c_1p}{a\sqrt{(a^2-f^2)}\sqrt{(a^2-g^2)}}.$$

Now, by (b), $dn=\dfrac{ada}{p}$, and hence

$$dv=-4\pi k_1\frac{a_1b_1c_1da}{\sqrt{(a^2-f^2)}\sqrt{(a^2-g^2)}}.$$

Integrating this, we have

$$v=-4\pi k_1a_1b_1c_1\int\frac{da}{\sqrt{(a^2-f^2)}\sqrt{(a^2-g^2)}}+C \qquad (e).$$

16. The two constants, k_1 and C, must be determined by the conditions $v=v_1$ when $a=a_1$, and $v=0$ when $a=\infty$; the latter of which must be fulfilled, in order that the expression found for v may be equal to $\displaystyle\iint\frac{k_1p_1d\omega_1^{\,2}}{r_1}$.

17. To reduce the expression for v to an elliptic function, let us assume

$$\left.\begin{aligned} a&=f\operatorname{cosec}\phi\\ a_1&=f\operatorname{cosec}\phi_1\end{aligned}\right\} \qquad (f),$$

which we may do with propriety if f be the greater of the two quantities f and g, since a is always greater than either of them, as we see from (d). On this assumption, equation (e) becomes

$$v=\frac{4\pi k_1a_1b_1c_1}{f}\int_0^{\phi}\frac{d\phi}{\sqrt{(1-c'^2\sin^2\phi)}}+C=\frac{4\pi k_1a_1b_1c_1}{f}F_{c'}\phi+C$$

where

$$c'=\frac{g}{f} \qquad (g).$$

18. Determining from this the values of C and k_1 by the conditions mentioned above, we find $C=0$, and

$$k_1=\frac{fv_1}{4\pi a_1 b_1 c_1 F_{c'}\phi_1} \qquad (h);$$

hence the expression for v becomes

$$v=v_1\frac{F_{c'}\phi}{F_{c'}\phi_1} \qquad (k).$$

19. The results which have been obtained may be stated as follows:—

If, in an infinite solid, the surface of an ellipsoid be retained at a constant temperature, the temperature of any point in the solid will be the same as that of any other point in the surface of an ellipsoid described from the same foci, and passing through that point; and the flux of heat at any point in the surface of this ellipsoid will be proportional to the perpendicular from the centre to a plane touching it at the point, and inversely proportional to the volume of the ellipsoid.

20. This case of the uniform motion of heat was first solved by Lamé, in his Mémoire on Isothermal Surfaces, in Liouville's *Journal de Mathématiques*, vol. ii. p. 147, by showing that a series of isothermal surfaces of the second order will satisfy the equation

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0,$$

provided they are all described from the same foci. The value which he finds for v agrees with (e), and he finds, for the flux of heat at any point, the expression

$$\frac{KA}{\sqrt{(\mu^2-\nu^2)}\sqrt{(\mu^2-\rho^2)}};$$

or, according to the notation which we have employed,

$$\frac{4\pi k_1 a_1 b_1 c_1}{\sqrt{(a^2-\nu^2)}\sqrt{(a^2-\rho^2)}},$$

where ν is the greater real semi-axis of the hyperboloid of one sheet, and ρ the real semi-axis of the hyperboloid of two sheets, described from the same foci as the original ellipsoid, and passing through the point considered. Hence a^2, ν^2, ρ^2 are the three roots of the equation

$$\frac{x^2}{u}+\frac{y^2}{u-f^2}+\frac{z^2}{u-g^2}=1,$$

or

$$u^3-(f^2+g^2+x^2+y^2+z^2)u^2+\{f^2g^2+(f^2+g^2)x^2+g^2y^2+f^2z^2\}u-f^2g^2x^2=0.$$

Hence $$a^2\nu^2\rho^2=f^2g^2x^2,$$

and $$a^2\nu^2+a^2\rho^2+\nu^2\rho^2=f^2g^2+(f^2+g^2)x^2+g^2y^2+f^2z^2.$$

Therefore,

$$\begin{aligned}(a^2-\nu^2)(a^2-\rho^2)&=a^4-a^2\nu^2-a^2\rho^2-\nu^2\rho^2+\frac{2a^2\nu^2\rho^2}{a^2}\\&=a^4-\{f^2g^2+(f^2+g^2)x^2+g^2y^2+f^2z^2\}+2f^2g^2\frac{x^2}{a^2}\\&=a^4-(a^2-b^2)(a^2-c^2)-(2a^2-b^2-c^2)x^2-(a^2-c^2)y^2\\&\qquad-(a^2-b^2)z^2+2(a^2-b^2)(a^2-c^2)\frac{x^2}{a^2}\\&=a^4-(a^2-b^2)(a^2-c^2)-(b^2+c^2)x^2-(a^2-c^2)y^2\\&\qquad-(a^2-b^2)z^2+2b^2c^2\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)\\&=a^4-(a^2-b^2)(a^2-c^2)-(b^2+c^2)x^2-(a^2+c^2)y^2-(a^2+b^2)z^2+2b^2c^2\\&=a^2b^2+a^2c^2+b^2c^2-\{(b^2+c^2)x^2+(a^2+c^2)y^2+(a^2+b^2)z^2\};\end{aligned}$$

which is readily shown, by substituting for $a^2b^2+a^2c^2+b^2c^2$ its equal $(a^2b^2+a^2c^2+b^2c^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$, to be equal to $\frac{a^2b^2c^2}{p^2}$. Hence the expression for $-\frac{dv}{dn}$, given above, becomes

$$-\frac{dv}{dn}=4\pi k_1\frac{a_1b_1c_1}{abc}p,$$

which agrees with (c).

Attraction of a Homogeneous Ellipsoid on a Point within or without it.

21. If, in (c), we put $k_1=\frac{da_1}{a_1}$, the value of $-\frac{dv}{dn}$ at any point will be the attraction on that point of a shell bounded by two similar concentric ellipsoids, whose semi-axes are

$$a_1,\ a_1\sqrt{(1-e^2)},\ a_1\sqrt{(1-e'^2)},$$

and $$a_1+da_1,\ (a_1+da_1)\sqrt{(1-e^2)},\ (a_1+da_1)\sqrt{(1-e'^2)},$$

where $$\left.\begin{aligned}a^2-b^2=a_1^2-b_1^2=a_1^2e^2\\ a^2-c^2=a_1^2-c_1^2=a_1^2e'^2\end{aligned}\right\}\qquad(1),$$
and

the density of the shell being unity. Now this attraction is in a normal drawn through the point attracted to the surface of the ellipsoid, whose semi-axes are a, b, c. If we call α, β, γ the angles which this normal makes with the co-ordinates x, y, z of the point attracted, we have

$$\cos\alpha = \frac{\dfrac{x}{a^2}}{\sqrt{\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}\right)}} = \frac{px}{a^2},$$

and similarly, $\cos\beta = \dfrac{py}{b^2}, \quad \cos\gamma = \dfrac{pz}{c^2}.$

Hence, calling dA, dB, dC the components of the attraction parallel to the axes of co-ordinates, we have, from (c),

$$\left.\begin{aligned} dA &= 4\pi x\frac{b_1c_1}{a^2bc}p^2da_1 \\ dB &= 4\pi y\frac{b_1c_1}{ab^2c}p^2da_1 \\ dC &= 4\pi z\frac{b_1c_1}{abc^2}p^2da_1 \end{aligned}\right\} \qquad (2).$$

22. The integrals of these expressions, between the limits $a_1 = 0$ and $a_1 = a_1'$, are the components of the attraction of an ellipsoid whose semi-axes are a_1', b_1', c_1', or $a_1', a_1'\sqrt{(1-e^2)}$, $a_1'\sqrt{(1-e'^2)}$, on the point (x, y, z). Now, by (1), we may express each of the quantities b, c, b_1, c_1, in terms of a and a_1, and the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{a^2-e^2a_1^2}+\frac{z^2}{a^2-e'^2a_1^2}=1 \qquad (3),$$

enables us to express either of the quantities a, a_1 in terms of the other. The simplest way, however, to integrate equations (2), will be to express each in terms of a third quantity,

$$u = \frac{a_1}{a} \qquad (4).$$

Eliminating a from (3), by means of this quantity, we have

$$a_1^2 = u^2x^2 + \frac{y^2}{u^{-2}-e^2} + \frac{z^2}{u^{-2}-e'^2}.$$

Hence $$a_1da_1 = \left\{ux^2 + \frac{u^{-3}y^2}{(u^{-2}-e^2)^2} + \frac{u^{-3}z^2}{(u^{-2}-e'^2)^2}\right\}du$$
$$= \left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)a_1^4u^{-3}du = a_1^4p^{-2}u^{-3}du.$$

Also, from (4), we have $a = \dfrac{a_1}{u}$; from which we find, by (1),

$b=\frac{a_1}{u}\sqrt{(1-e^2u^2)}$, $c=\frac{a_1}{u}\sqrt{(1-e'^2u^2)}$. By (1) also, $b_1=a_1\sqrt{(1-e^2)}$, $c_1=a_1\sqrt{(1-e'^2)}$. Making these substitutions in (2), and integrating, we have, calling a' the value of a, when $a_1=a_1'$,

$$\left.\begin{aligned}
A&=4\pi x\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}}\\
B&=4\pi y\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}}\\
C&=4\pi z\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}}
\end{aligned}\right\}\quad(5).$$

23. If the point attracted be within the ellipsoid, the attraction of all the similar concentric shells without the point will be nothing; and hence the superior limit of u will be the value of $\frac{a_1}{a}$ at the surface of an ellipsoid, similar to the given one, and passing through the point attracted.

Now, in this case, $a_1=a$, since a is one of the semi-axes of an ellipsoid passing through the point attracted, and having the same foci as another ellipsoid (passing through the same point) whose corresponding semi-axis is a_1. Hence, for an interior point, we have

$$\left.\begin{aligned}
A&=4\pi x\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}}\\
B&=4\pi y\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}}\\
C&=4\pi z\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}}
\end{aligned}\right\}\quad(6).$$

24. These are the known expressions for the attraction of an ellipsoid on a point within it. Equations (5) agree with the expressions given in the Supplement to Liv. v. of Pontécoulant's *Théorie Analytique du Système du Monde*, where they are found by direct integration, by a method discovered by Poisson. They may also be readily deduced from equations (6) by Ivory's Theorem. Or, on the other hand, by a comparison of them, after reducing the limits of the integrals to 0 and 1, by substituting $\frac{a_1'}{a'}v$ for u, with equation (6), Ivory's Theorem may be readily demonstrated.

II. ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

I.—ON THE ELEMENTARY LAWS OF STATICAL ELECTRICITY.*

[From *Cambridge and Dublin Mathematical Journal*, Nov. 1845. Reprinted *Philosophical Magazine*, 1854, second half-year, with additional Notes of date March 1854.]

25. The elementary laws which regulate the distribution of electricity on conducting bodies have been determined by means of direct experiments, by Coulomb, and in the form he has given them, which is independent of any hypothesis,† they have long been considered as rigorously established. The problem of the distribution of electricity in equilibrium on a conductor of any form was thus brought within the province of mathematical analysis; but the solution, even in the simplest cases, presented so much difficulty that Coulomb, after having investigated it experimentally for bodies of various forms, could only compare his measurements with the results of his theory by very rude processes of approximation. Without, however, giving rigorous solutions in particular cases, he examined the general problem with great care, and left nothing indefinite in the conditions to be satisfied, so that it was entirely by analytical difficulties that he was stopped. As an example of the

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* This paper is a translation (with considerable additions) of one which appeared in Liouville's *Journal de Mathématiques*, 1845, p. 209.

† Coulomb has expressed his theory in such a manner that it can only be attacked in the way of proving his experimental results to be inaccurate. This is shown in the following remarkable passage in his sixth memoir, which follows a short discussion of some of the physical ideas then commonly held with reference to electricity. "*Je préviens pour mettre la théorie qui va suivre à l'abri de toute dispute systématique, que dàns la supposition des deux fluides électriques, je n'ai d'autre intention que de présenter avec le moins d'élémens possible, les resultats du calcul et de l'expérience, et non d'indiquer les véritables causes de l'électricité. Je renverrai, à la fin de mon travail sur l'électricité, l'examen des principaux systèmes auxquels les phénomènes électriques ont donné naissance.*"—Histoire de l'Académie, 1788, p. 673.

success of his theoretical investigations, we may refer to the well-known demonstration of the theorem (usually attributed to Laplace) relative to the repulsion exercised by a charged conductor on a point near its surface.[*]

The memoirs of Poisson, on the mathematical theory, contain the analytical determination of the distribution of electricity on two conducting spheres placed near one another, the solution being worked out in numbers in the case of two equal spheres in contact, which had been investigated experimentally by Coulomb (as well as in another case, not examined by Coulomb, which is given as a specimen of the numerical results that may be deduced from the formulæ). The calculated ratios of the intensities at different points of the surface he is therefore enabled to compare with Coulomb's measurements, and he finds an agreement which is quite as close as could be expected, when we consider the excessively difficult and precarious nature of quantitative experiments in electricity: but the most remarkable confirmation of the theory from these researches is the entire agreement of the principal features, even in some very singular phenomena, of the experimental results with the theoretical deductions. For a complete account of the experiments we must refer to Coulomb's fifth memoir (*Histoire de l'Académie*, 1787), and for the mathematical investigations to the first and second memoirs of Poisson (*Mémoires de l'Institut*, 1811), or to the treatise on Electricity in the *Encyclopædia Metropolitana*, where the substance of Poisson's first memoir is given.

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[*] This theorem may be stated as follows:—Let A be a closed surface of any form, and let matter, attracting inversely as the square of the distance, be so distributed over it that the resultant attraction on an interior point is nothing: the resultant attraction on an exterior point, indefinitely near any part of the surface, will be perpendicular to the surface and equal to $4\pi\rho$, if $\rho\omega$ be the quantity of matter on an element ω of the surface in the neighbourhood of the point. Coulomb's demonstration of this theorem may be found in a preceding paper in the *Mathematical Journal*, vol. iii. p. 74 (above, I. 7). He gives it himself, in his sixth memoir on Electricity (*Histoire de l'Académie*, 1788, p. 677), in connexion with an investigation of the theory of the proof plane in which, by an error that is readily rectified, he arrives at the result that a small insulated conducting disc, put in contact with an electrified conductor at any point, and then removed, carries with it as much electricity as lies on an element of the conductor at that point equal in area to the two faces of the disc; the quantity actually removed being only half of this. This result, however, does not at all affect the experimental use which he makes of the proof plane, which is merely to find the ratios of the intensities at different points of a charged conductor. As the complete theory of this valuable instrument has not, so far as I am aware, been given in any English work, I annex the following remarkably clear account of it, which is extracted from Pouillet's *Traité de Physique*:—"Quand le plan d'épreuve est tangent à une surface, il se confond avec l'élément qu'il touche, il prend en quelque sorte sa place relativement à l'électricité, ou plutôt il devient lui-même l'élément sur lequel la fluide se répand; ainsi, quand on retire ce plan, on fait la même chose que si l'on avait découpé sur la surface un élément de même épaisseur et de même étendue que lui, et qu'on l'eût enlevé pour le porter dans la balance sans qu'il perdît rien de l'électricité qui le

The mathematical theory received by far the most complete development which it has hitherto obtained, in Green's *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*,[*] in which a series of general theorems were demonstrated, and many interesting applications made to actual problems.[†]

Of late years some distinguished experimentalists have begun to doubt the truth of the laws established by Coulomb, and have made extensive researches with a view to discover the laws of certain phenomena which they considered incompatible with his theory. The most remarkable works of this kind

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couvre; une fois séparé de la surface, cet élément n'aurait plus dans ses différents points qu'une épaisseur électrique moitié moindre, puisque la fluide devrait se répandre pour en couvrir les deux faces. Ce principe posé, l'expérience n'exige plus que de l'habitude et de la dextérité: après avoir touché un point de la surface avec le plan d'épreuve, on l'apporte dans la balance, où il partage son électricité avec le disque de l'aiguille qui lui est égale, et l'on observe la force de torsion à une distance connue. On répète la même experience en touchant un autre point, et le rapport des forces de torsion est le rapport des repulsions électriques; on en prend la racine carrée pour avoir le rapport des épaisseurs. Ainsi le génie de Coulomb a donné en même temps aux mathématiciens la loi fondamentale suivant laquelle la matière électrique s'attire et se repousse; et aux physiciens une balance nouvelle, et des principes d'éxpérience au moyen desquels ils peuvent en quelque sorte sonder l'épaisseur de l'électricité sur tous les corps, et determiner les pressions qu'elle exerce sur les obstacles qui l'arrêtent."

To this explanation it should be added that, when the proof plane is still very near the body to which it has been applied, the effect of mutual influence is such as to make the intensity be insensible at every point of the disc on the side next the conductor, and at each point of the conductor which is *under* the disc. It is only when the disc is removed to a considerable distance that the electricity spreads itself symmetrically on its two faces, and that the intensity at the point of the conductor to which it was applied, recovers its original value. It was the omission of this consideration that caused Coulomb to fall into the error alluded to above.

[*] Nottingham, 1828.

[†] This memoir of Green's has been unfortunately very little known, either in this country or on the Continent. Some of the principal theorems

have been undertaken independently by Mr. Snow Harris and Mr. Faraday, and in their memoirs, published in the *Philosophical Transactions*, we find detailed accounts of their researches. All the experiments, however, which they have made, having direct reference to the distribution of electricity in equilibrium, are, I think, in full accordance with the laws of Coulomb, and must therefore, instead of objections to his theory, be considered as confirming it. As, however, many have believed Coulomb's theory to be overturned by these investigations, and as others have at least been led to entertain doubts as to its certainty or accuracy, the following attempt to explain the apparent difficulties is made the subject of the first of a series of papers in which various parts of the mathematical theory of electricity, and corresponding problems in the theories of magnetism and heat, will be considered.

26. We may commence by examining some experimental results published in Mr. Harris's first memoir *On the Elementary Laws of Electricity*.* After describing the instruments employed in his researches, Mr. Harris gives the details of some experiments with reference to the attraction exercised by an insulated electrified body on an uninsulated conductor placed in its neighbourhood. The first result which he an-

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in it have been re-discovered within the last few years, and published in the following works:—

Comptes Rendus for Feb. 11th, 1839, where part of the series of theorems is announced without demonstration, by Chasles.

Gauss's memoir on "General Theorems relating to Attractive and Repulsive Forces, varying inversely as the square of the distance," in the *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1839, *Leipsic*, 1840. (Translations of this paper have been published in *Taylor's Scientific Memoirs* for April 1842, and in the Numbers of *Liouville's Journal* for July and August 1842.)

Mathematical Journal, vol. iii., Feb. 1842, in a paper "On the Uniform Motion of Heat, etc." (I. above).

Additions to the Connaissance des Tems for 1845 (published June 1842), where Chasles supplies demonstrations of the theorems which he had previously announced.

I should add that it was not till the beginning of the present year (1845) that I succeeded in meeting with Green's Essay. The allusion made to his name with reference to the word "potential" (*Mathematical Journal*, vol. iii. p. 190), was taken from a memoir of Murphy's, "On definite Integrals with Physical Applications," in the *Cambridge Transactions*, where a mistaken definition of that term, as used by Green, is given.

* *Philosophical Transactions*, 1834.

nounces is that, when other circumstances remain the same, the attraction varies as the square of the quantity of electricity with which the insulated body is charged. It is readily seen, as was first remarked by Dr. Whewell in his *Report on the Theories of Electricity, etc.*,* that this is a rigorous deduction from the mathematical theory, following from the fact that the quantity of electricity induced upon the uninsulated body is proportional to the charge on the electrified body by which it is attracted.

27. The remaining results have reference to the force of attraction at different distances, and with bodies of different forms opposed. As these are generally very irregular (such as "plane circular areas backed by small cones"), we should not, according to Coulomb's theory, expect any very simple laws, such as Mr. Harris discovers, to be rigorously true. Accordingly, though they are announced by him without restriction, we must examine whether the experiments from which they have been deduced are of a sufficiently comprehensive character to lead to any general conclusions with respect to electrical action. Now, in the first place, we find that in all of them the attraction is "independent of the form of the unopposed parts" of the bodies, which will be the case only when the intensity of the induced electricity on the unopposed parts of the uninsulated body is insensible. According to the mathematical theory, and according to Mr. Faraday's researches "on induction in curved lines," which will be referred to below, the intensity never absolutely vanishes at any point of the uninsulated body: but it is readily seen that in the case of Mr. Harris's experiments, it will be so slight on the unopposed portions that it could not be perceived without experiments of a very refined nature, such as might be made by the proof plane of Coulomb, which is in fact, with a slight modification, the instrument employed by Mr. Faraday in the investigation. Now to the degree of approximation to which the intensity on the unopposed parts may be neglected, the laws observed by Mr. Harris when the opposed surfaces are plane may be readily deduced from the mathematical theory. Thus let v be the

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* *British Association Report for* 1837.

potential in the interior of the charged body, A; a quantity which will depend solely on the state of the interior coating of the battery with which in Mr. Harris's experiments A is connected, and will therefore be sensibly constant for different positions of A relative to the uninsulated opposed body, B. Let a be the distance between the plane opposed faces of A and B, and let S be the area of the opposed parts of these faces, which will in general be the area of the smaller, if they be unequal. When the distance a is so small that we may entirely neglect the intensity on all the unopposed parts of the bodies, it is readily shown from the mathematical theory that (since the difference of the potentials at the surfaces of A and B is v) the intensity of the electricity produced by induction at any point of the portion of the surface of B which is opposed to A, is $\frac{v}{4\pi a}$. Hence the attraction on any small element ω, of the portion S of the surface of B, will be in a direction perpendicular to the plane and equal to $2\pi\left(\frac{v}{4\pi a}\right)^2$.* Hence the whole attraction on B is

$$\frac{v^2 S}{8\pi a^2}.$$

This formula expresses all the laws stated by Mr. Harris as results of his experiments in the case when the opposed surfaces are plane.

28. When the opposed surfaces are curved, for instance when A and B are equal spheres, we can make no approximation analogous to that which has led us to so simple an expression in the case of opposed planes; and we find accordingly that no such simple law for the attraction in this case has been announced by Mr. Harris. He has, however, found that it is expressed with tolerable accuracy by the formula

$$F = \frac{k}{c(c-2a)},$$

where c is the distance between the centres of the spheres, a the radius of each, k a constant, which will depend on a and on the charge of the battery with which A is in communication. Though, however, this formula may give results which do not differ very much from observation within a limited range of distances, it cannot, according to any theory, be considered as expressing the physical law of the phenomenon. For, according to it, when the balls are very distant, F ultimately varies as $\frac{1}{c^2}$. Now it is clear that the law of force must ultimately become the inverse cube of the distance, since the quantity of electricity induced upon B will be ultimately in the inverse ratio of the distance, and the attraction between the balls as the product of the quantities of electricity directly, and as the square of the distance inversely, and hence the formula given by Mr. Harris cannot express the law of force when the balls are very distant. In the experiments by which his formula is tested, the force of attraction is measured by means of an ordinary balance and weights: the only comparison of results which he publishes is transcribed in the following table:—

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* See VII. below.

Dist. of Centres.	Measured Force in Grains.	Values of $\frac{15c_1(c_1-2)}{c(c-2)}$.
$c_1=2·3$	15	15
$c_2=2·5$	8·25 +	8·28
$c_3=2·8$	4·6 +	4·62
$c_4=3·0$	3·5 −	3·45

29. From this table we see that the formula is verified in three cases to the extent of accuracy of the experiments. Comparisons extended to a much wider range of distances would be required to establish it, and it would be necessary to take precautions to prevent the experimental results from being influenced by disturbing causes. In the experiments made by Mr. Harris, we find that no precautions have been taken to avoid the disturbing influence of extraneous conductors, which, according to the descriptions and drawings he gives of his instruments, seem to exist very abundantly in the neighbourhood of the bodies operated upon, being partly metal in connexion with the insulated system with which the body A communicates, and partly uninsulated metal, in the fixed parts of the electro-

meter, and in the movable parts by which B is supported. The general effect produced by the presence of such bodies in disturbing the observed law of force, must be to make it diminish less rapidly with the distance when A and B are separated by a considerable interval: and it is probably owing, at least in part, to such disturbing causes that Mr. Harris's results nearly agree, as far as they go, with a formula which would ultimately give for the law of force the inverse square of the distance between A and B, instead of the inverse cube.

30. The determination by the mathematical theory of the attraction or repulsion between two electrified conducting spheres has not hitherto, so far as I am aware, been attempted, and would present considerable difficulty by means of the formulæ ordinarily given for such problems. It may, however, very readily be effected by means of a general theorem on the attraction between electrified conductors, which will be given in a subsequent paper.* Thus, if $F(c)$ be the force of attraction, corresponding to the distance c between the centres, in the particular case when the two spheres are equal (the radius of each being unity), and the potential in the interior of one of them is nothing (as will be the case when the body is uninsulated), the potential in the interior of the other being v, I have found the following formulæ, which express $F(c)$ by a converging series:—

$$\text{(A)} \qquad F(c)=v^2c\left(\frac{P_1}{Q_1^{\,2}}+\frac{P_2}{Q_2^{\,2}}+\frac{P_3}{Q_3^{\,2}}+\text{etc.}\right), \text{ where}$$

$$\text{(B)}, \qquad \begin{cases} Q_1 \;\;\;\, =c^2-1, \\ Q_2 \;\;\;\, =(c^2-2)Q_1-1, \\ Q_{n+2}=(c^2-2)Q_{n+1}-Q_n. \end{cases}$$

$$\text{(C)}, \qquad \begin{cases} P_1 \;\;\;\, =1, \\ P_2 \;\;\;\, =2c^2-3, \\ P_{n+2}=(c^2-2)P_{n+1}+(Q_{n+1}-P_n). \end{cases}$$

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* [*Note added March* 1854.—The enunciation of the "general theorem" alluded to, the investigation founded on it, by which the author first arrived at the conclusion made use of here, and another demonstration of the same conclusion, founded on the method of electrical images, and strictly synthetical in its character, are published, with comprehensive numerical results, in the *Philosophical Magazine* for April 1853.]

31. These formulæ enable us to calculate Q_1, Q_2, Q_3, Q_4, etc., and then P_1, P_2, P_3, P_4, etc., successively, by a simple and uniform arithmetical process, for any particular value of c, I have thus calculated the values of $\frac{F(c)}{v^2}$ in five cases, the first four of which are those examined by Mr. Harris, and have obtained the following results, each of which is true to five places of decimals:—

c.	$v^{-2}F(c)$.
2·3	0·32926
2·5	0·17423
2·8	0·09168
3·0	0·06592
4·0	0·02075

32. To compare these with Mr. Harris's measurements, we may calculate the value of the potential in his battery, during the observations, by means of his first result, and thence find the attraction for the other three cases by means of the calculated values of $v^{-2}F(c)$. Thus we have $v^{-2} \times 15 = {·}3293$, which gives

$$v^2 = 45{·}56,$$

and hence

$$F(2{·}5) = 7{·}94,$$

$$F(2{·}8) = 4{·}18,$$

$$F(3) = 3{·}00.$$

These numbers differ considerably from Mr. Harris's results, but in the direction indicated by the considerations mentioned above.

33. The most important part of the researches of Mr. Harris is that in which he investigates the insulating power of air of different densities. The result at which he arrives is, that the intensity necessary to produce a spark depends solely on the density of the air, and not otherwise on the pressure or temperature. He thus shows that the conducting power of flame, of heated bodies, and of a vacuum, are due solely to the rarefaction of the air in each case. He also shows that the intensities necessary to produce a spark are in the simple ratios of the densities of the air.

34. In a subsequent memoir, by the same author,* we find additional experiments on the elementary principles of the theory of electricity. The first series which is described, was made for the purpose of testing the truth of Coulomb's law, that the repulsion of two similarly charged points is inversely as the square of the distance, and directly as the product of the masses. In experiments of this kind in which accurate quantitative results are aimed at, many precautions are necessary. Thus all conducting bodies, except those operated upon, must be placed beyond the reach of influence, and the distance between the repelling bodies must be considerable with reference to their linear dimensions, so that the distribution of electricity on each may be uninfluenced by the presence of the other. Also the bodies should be spheres, so that the attraction may be the same as if the whole electricity of each were collected at its centre; and the distance to be measured will then be the distance between the centres. These conditions have been expressly mentioned by Coulomb, and they have been fulfilled, as far as possible, in his researches, as we see by the descriptions of the experiments made, which we find in his memoirs. He has thus arrived by direct measurement at the law, which we know by a mathematical demonstration,† founded upon independent experiments, to be the rigorous law of nature, for electrical action. None of these precautions, however, have been taken in the experiments described in Mr. Harris's

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* *Philosophical Transactions*, 1836.

† See Murphy's *Electricity*, p. 41, or Pratt's *Mechanics*, Art. 154.

[*Note added March* 1854.—Cavendish demonstrates mathematically that if the law of force be any other than the inverse square of the distance, electricity could not rest in equilibrium on the surface of a conductor. But experiment has shown that electricity does rest at the surface of a conductor. Hence the law of force must be the inverse square of the distance. Cavendish considered the second proposition as highly probable, but had not experimental evidence to support this opinion, in his published work (An attempt to explain the phænomena of Electricity by means of an Elastic Fluid). Since his time, the most perfect experimental evidence has been obtained that electricity resides at the surface of a conductor; in such facts, for instance, as the perfect equivalence in all electro-statical relations of a hollow metallic conductor of ever so thin substance, or of a gilt non-conductor (possessing a conducting film of not more than $\frac{1}{500000}$ of an inch thick) and a solid conductor of the same external form and dimensions; the minor premise of his syllogism is thus demonstrated, and the conclusion is therefore established.]

memoir, and the results are accordingly unavailable for the accurate *quantitative* verification of any law, on account of the numerous unknown disturbing circumstances by which they are affected. The phenomena which he observes, however, afford *qualitative* illustrations of the mathematical theory of a very interesting nature, as may be seen from the following examples of his results :—

(*a.*) When the distance between the bodies is great with reference to their linear dimensions, the repulsion is inversely as the square of the distance, and directly as the product of the masses.

(*b.*) When the distance is small, the action becomes apparently irregular. Thus if the quantities of electricity on the two bodies be equal, the force, which is always of repulsion, does not increase so rapidly when the bodies approach, as if it followed the law of the inverse square of the distance.

(*c.*) If the charges be unequal, the repulsion ceases at a certain distance, and at all smaller distances there is attraction between the bodies.

35. These results are, with all their peculiarities, in full accordance with the theory of Coulomb, which indicates that, if the quantities of electricity be equal, and the bodies equal and similar, there will be repulsion in every position : but if there be any difference, however small, between the charges, the repulsion will necessarily cease, and attraction commence, before contact takes place, when one body is made to approach the other. Unless, however, the difference of the charges be sufficiently considerable, a spark may pass between the bodies, and render the charges equal, before attraction commences. In Mr. Harris's experiments, in which the bodies seem to have been nearly oblate spheroids, the attraction is generally sensible before the distance is small enough to allow a spark to pass, if the charge on one be double of that on the other.

Mr. Harris next proceeds to investigate the theory of the proof plane, and to examine whether it can be considered as indicating with certainty the intensity of electricity at any part of a charged body, and, principally from an experiment made on a charged non-conductor (a hollow sphere of glass), comes to a negative conclusion. It should be remembered, however, that, the proof plane having never been applied to determine the intensity at points of the surface of a charged non-conductor, such conclusions in no way interfere with adopted ideas. Since there can be no manner of doubt as to the theory of this valuable instrument, as we find it explained by M. Pouillet,* nor as to the experimental use of it made by Coulomb, it is unnecessary to enter more at length on the subject here.

36. Mr. Faraday's researches on electrostatical induction, which are published in a memoir forming the eleventh series of his Experimental Researches in Electricity, were undertaken with a view to test an idea which he had long possessed, that the forces of attraction and repulsion exercised by free electricity, are not the resultant of actions exercised at a distance, but are propagated by means of molecular action among the contiguous particles of the insulating medium surrounding the electrified bodies, which he therefore calls the *dielectric.* By this idea he has been led to some very remarkable views upon induction, or, in fact, upon electrical action in general. As it is impossible that the phenomena observed by Faraday can be incompatible with the results of experiment which constitute Coulomb's theory, it is to be expected that the difference of his ideas from those of Coulomb must arise solely from a different method of stating, and interpreting physically, the same laws: and farther, it may, I think, be shown that either method of viewing the subject, when carried sufficiently far, may be made the foundation of a mathematical theory which would lead to the elementary principles of the other as consequences. This theory would accordingly be the expression of the ultimate law of the phenomena, independently of any physical hypothesis we might, from other circumstances, be led to adopt. That there are necessarily two distinct elementary ways of viewing the theory of electricity, may be seen from the following considerations, founded on the principles developed in a previous paper in this Journal.†

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* See foot-note on § 23.

† On the Uniform Motion of Heat, and its Connexion with the Mathematical Theory of Electricity (I. above).

37. Corresponding to every problem relative to the distribution of electricity on conductors, or to forces of attraction and repulsion exercised by electrified bodies, there is a problem in the uniform motion of heat which presents the same analytical conditions, and which, therefore, considered mathematically, is the same problem. Thus, let a conductor A, charged with a given quantity of electricity, be insulated in a hollow conducting shell, B, which we may suppose to be uninsulated. According to the mathematical theory, an equal quantity of electricity of the contrary kind will be attracted to the interior surface of B (or the surface of B, as we may call it to avoid circumlocution), and the distribution of this charge, and of the charge on A, will take place so that the resultant attraction at any point of each surface may be in the direction of the normal. This condition being satisfied, it will follow that there is no attraction on any point within A, or without the surface of B, that is, on any point within either of the conducting bodies. The most convenient mathematical expression for the condition of equilibrium, is that the potential at any point P[*] must have a constant value when P is on the surface of A, and the value nothing when P is on the surface of B; and it will follow from this that the potential will have the same constant value for any point within A, and will be equal to nothing for any point without the surface of B.

If A be subject to the influence of any uninsulated conductors, we must consider such bodies as belonging to the shell in which A is contained, and their surfaces as forming part of the surface of B: in such cases this surface will generally be the interior surface of the walls of the room in which A is contained, and of all uninsulated conductors in the room. If, however, we have to consider the case in which A is subject to no external influence, we must suppose every part of the surface of B to be very far from A. The most general problem we can contemplate in electricity (exclusively of the case in which the insulating medium is heterogeneous, and exercises a special action, which will be alluded to below), is to determine the potential at any point when A, instead of being a single conductor, is a group of separate insulated conductors charged to different degrees, and when there are non-conductors electrified in a given manner, placed in the insulating medium, in the neighbourhood. The conditions of equilibrium will still be that the potential at each surface due to all the free electricity must be constant, and the theorems stated above will still be true: thus the attraction will be nothing in the interior of each portion of A, and without the surface of B; and the whole quantity of induced electricity on the latter surface will be the algebraic sum of the charges of all the interior bodies with its sign changed. When the potential due to such a system is determined for every point, the component of the resultant force at any point P, in any direction PL, may be found by differentiation, being the limit of the difference between the values of the potential at P, and at a point Q, in PL, divided by PQ, when Q moves up towards and ultimately coincides with P, and the direction of the force, on a *negative* particle, being that in which the potential increases. By Coulomb's theorem, the intensity at any point in one of the conducting surfaces is equal to the attraction (on a negative unit) at that point, divided by 4π.

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[*] The term used by Green for the sum of the quotients obtained by dividing the product of each element of the surfaces of A and B, and its electrical intensity, by its distance from P.

38. Now if we wish to consider the corresponding problem in the theory of heat, we must suppose the space between A and B, instead of being filled with a dielectric medium (that is a non-conductor for electricity), to be occupied by any homogeneous solid body, and sources of heat or cold to be so distributed over the terminating surfaces, or the interior surface of B and the surface of A, that the permanent temperature at the first surface may be zero, and at the second shall have a certain constant value, the same as that of the *potential* in the case of electricity. If A consist of different isolated portions, the temperature at the surface of each will have a constant value, which is not necessarily the same for the different portions. The problem of *distributing sources of heat, according to these conditions*, is mathematically identical with the problem of distributing *electricity in equilibrium* on the surfaces of A and B. In the case of heat, the *permanent temperature* at any point replaces the *potential* at the corresponding point in the

electrical system, and consequently the *resultant flux of heat* replaces the *resultant attraction* of the electrified bodies, in direction and magnitude. The problem in each case is determinate, and we may therefore employ the elementary principles of one theory, as theorems, relative to the other. Thus, in the paper in which these considerations are developed, Coulomb's fundamental theorem relative to electricity is applied to the theory of heat; and self-evident propositions in the latter theory are made the foundation of Green's theorems in electricity.[*] Now the laws of motion for heat which Fourier lays down in his *Théorie Analytique de la Chaleur*, are of that simple elementary kind which constitute a mathematical theory properly so called; and therefore, when we find corresponding laws to be true for the phenomena presented by electrified bodies, we may make them the foundation of the mathematical theory of electricity: and this may be done if we consider them merely as actual truths, without adopting any physical hypothesis, although the idea they naturally suggest is that of the propagation of some effect by means of the mutual action of contiguous particles; just as Coulomb, although his laws naturally suggest the idea of material particles attracting or repelling one another at a distance, most carefully avoids making this a *physical hypothesis*, and confines himself to the consideration of the mechanical effects which he observes and their necessary consequences.[†]

39. All the views which Faraday has brought forward, and illustrated or demonstrated by experiment, lead to this method of establishing the mathematical theory, and, as far as the analysis is concerned, it would, in most *general* propositions, be even more simple, if possible, than that of Coulomb. (Of course the analysis of *particular* problems would be identical in the two methods.) It is thus that Faraday arrives at a knowledge of some of the most important of the general

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[*] It was not until some time after that paper was published, that I was able to add the direct analytical demonstrations of the theorems, which are given in the papers on "General Propositions in the Theory of Attraction," *Camb. Math. Jour.*, vol. iii. pp. 189, 201 (XII. below), and which I have since found are the same as those originally given by Green.

[†] See first foot-note on § 25.

theorems, which, from their nature, seemed destined never to be perceived except as mathematical truths. Thus, in his theory, the following proposition is an elementary principle :—Let any portion α of the surface of A be projected on B, by means of lines (which will be in general curved) possessing the property that the resultant electrical force at any point of each of them is in the direction of the tangent: the quantity of electricity produced by induction on this projection is equal to the quantity of the opposite kind of electricity on α.[*] The lines thus defined are what Faraday calls the "curved lines of inductive action." For a detailed account of the experiments by which these phenomena are investigated, reference must be made to Mr. Faraday's own memoirs, published in the *Philosophical Transactions*, and in a separate form in his *Experimental Researches*.

*40. The hypothesis adopted by Faraday, of the *propagation* of inductive action, naturally led him to the idea that its effects may be in some degree dependent upon the nature of the insulating medium or dielectric, by which, according to this view, it is transmitted. In the second part of his memoir he describes a series of researches instituted to put this to the test of experiment, and arrives at the following conclusions :—

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[*] This theorem may be proved as follows :—

Let S be any closed surface, containing no part of the electrified bodies within it, which we may conceive to be described between A and B; let P be the component in the direction of the normal, of the resultant force at any point of the surface S, and let ds be an element of the surface at the same point. Then it may be easily proved (see *Camb. Math. Jour.*, vol. iii. p. 204) that

$$\iint P ds = 0 \qquad (a),$$

the integrations being extended over the entire surface. Now let S be supposed to consist of three parts; the portion α, of the surface of A; its projection β, on the interior surface of B; and the surface generated by the curved lines of projection. The value of P at each point of the latter portion of S will be nothing, since the tangent at any point of a line of projection is the direction of the force. Hence, if $[\iint P ds]$ and $(\iint P ds)$ denote the values of $\iint P ds$, for the portions α and β of S, the equation (a) becomes

$$[\iint P ds] + (\iint P ds) = 0.$$

But if ρ be the intensity of the distribution on the surface A or B, at any point, we have, by Coulomb's theorem,

$$\rho = \frac{P}{4\pi}.$$

Hence $$[\iint \rho ds] + (\iint \rho ds) = 0,$$

which is the theorem quoted in the text.

41. If the dielectric be air, the inductive action is quite independent of its density or temperature (which, as Mr. Faraday remarks, agrees perfectly with previous results obtained by Mr. Harris); and in general, if the dielectric be any gas or vapour capable of insulating a charge, the inductive action is invariable. Hence he concludes that "*all gases have the same power of*, or *capacity for*, sustaining induction through them (which might have been expected when it was found that no variation of density or pressure produced any effect)."

When the dielectric is solid, the induction is greater than through air, and varies according to the nature of the substance. Numbers which measure the "specific inductive capacities" of the dielectrics employed (sulphur, shell lac, glass, etc.) are deduced from the experiments.

42. To express these results in the language of the mathematical theory, let us recur to the supposition of a body, *A*, charged with a given quantity of electricity, and insulated in the interior of a closed conducting shell, *B*. The potential of the system at the interior surface of *B*, and at every point without this surface, will be nothing; at the surface and in the interior of *A* it will have a constant value, which will depend on the form, magnitude, and relative position of the surfaces *A* and *B*, on the quantity of electricity on *A*, and, according to Faraday's discovery, on the *dielectric power* of the insulating medium which fills the space between *A* and *B*. If this be gaseous, neither its nature nor its state as to temperature, pressure, or density will affect the value of the potential in *A*; but if it be a solid substance, such as sulphur or shell lac, the value of the potential will be less than when the space is occupied by air, and will vary with the nature of the insulating solid.

43. The result in the case of a gaseous dielectric is what would follow from Coulomb's theory, if we consider gases to be quite impermeable to electricity, and to be entirely unaffected by electrical influence. The phenomena observed with solid dielectrics, which agree with the circumstance observed by Nicholson, that the *dissimulating power* of a Leyden phial depends on the nature of the glass of which it is made, as well as on its thickness, have been by some attributed to a slight degree of conducting power, or of penetrability, pos-

sessed by solid insulators. This explanation, however, seems to be very insufficient; and besides, Faraday has estimated the nature of the effects of imperfect insulation by independent experiments, and has established, in what seems to be a very satisfactory manner, the existence of a peculiar action in the interior of solid insulators when subjected to electrical influence. As far as can be gathered from the experiments which have yet been made, it seems probable that a dielectric, subjected to electrical influence, becomes excited in such a manner that every portion of it, however small, possesses *polarity* exactly analogous to the magnetic polarity induced in the substance of a piece of soft iron under the influence of a magnet. By means of a certain hypothesis regarding the nature of magnetic action,[*] Poisson has investigated the mathematical laws of the distribution of magnetism, and of magnetic attractions and repulsions. These laws seem to represent in the most general manner the state of a body polarized by influence, and therefore, without adopting any particular mechanical hypothesis, we may make use of them to form a mathematical theory of electrical influence in dielectrics, the truth of which can only be established by a rigorous comparison of its results with experiment.

44. Let us therefore consider what would be the effect, according to this theory, which would be produced by the presence of a solid dielectric, C, placed in the space between A and B, the rest of which is occupied by air. The action of C, when excited by the influence of the electricities on A and B, may (as Poisson has shown for magnetism) be represented, whether

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[*] Faraday adopts the corresponding hypothesis to explain the action of a solid dielectric, which he states thus :—"If the space round a charged globe were filled with a mixture of an insulating dielectric, as oil of turpentine or air, and small globular conductors, as shot, the latter being at a little distance from each other, so as to be insulated, then these in their condition and action exactly resemble what I consider to be the condition and action of the particles of the insulating dielectric itself. If the globe were charged, these little conductors would all be polar; if the globe were discharged, they would all return to their normal state, to be polarized again upon the recharging of the globe."—(*Experimental Researches*, § 1679.) The results of the mathematical analysis of such an action are given in the text. It may be added that the value of the coefficient k will differ sensibly from unity if the volume occupied by the small conducting balls bear a finite ratio to that occupied by the insulating medium.

on points within or without *C*, by a certain distribution of positive electricity on one portion of the surface of *C*, and of an equal quantity of negative electricity on the remainder. The condition necessary and sufficient for determining this distribution may (as can be shown from Poisson's analysis) be expressed as follows. Let *R* be the resultant force on a point *P* without *C*, and *R'* on a point *P'* within *C*, due to the electrified surfaces *A* and *B*, and to the imagined distribution on *C*. If *P* and *P'* be taken infinitely near one another, and consequently each infinitely near the surface of *C*, the component of *R'* in the direction of the normal must bear to the component of *R* in the same direction a constant ratio $\left(\frac{1}{k}\right)$ depending on the capacity for dielectric induction of the matter of *C*.* The components of *R* and *R'* in the tangent plane will of course be equal and in the same direction, and, if ρ be the intensity of the imagined distribution on the surface of *C*, in the neighbourhood of *P* and *P'*, the difference of the normal components will be $4\pi\rho$, as is evident from Coulomb's theorem, referred to above.

45. Let us now suppose *C* to be a shell surrounding *A*, and let *S* and *S'*, its interior and exterior surfaces, be *surfaces of equilibrium* in the system of forces due to the action of *A* and *B*, and of the polarity of *C*. It may be shown that the same surfaces *S*, *S'*, would necessarily be surfaces of equilibrium, if *C* were removed and the whole space were filled with air; and consequently, that the whole series of surfaces of equi-

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* From this it follows that, in the case of heat, *C* must be replaced by a body whose conducting power is *k* times as great as that of the matter occupying the remainder of the space between *A* and *B*.

[*Note added March* 1854.—The same demonstration, of course, is applicable to the influence of a piece of soft iron, or other "paramagnetic" (*i.e.*, substance of ferro-magnetic inductive capacity), or to the reverse influence of a diamagnetic on the magnetic force in any locality near a magnet in which it can be placed, and shows that the lines of magnetic force will be altered by it precisely as the lines of motion of heat in corresponding thermal circumstances would be altered by introducing a body of greater or of less conducting power for heat. Hence we see how strict is the foundation for an analogy on which the *conducting power of a magnetic medium for lines of force* may be spoken of, and we have a perfect explanation of the condensing action of a paramagnetic, and the repulsive effect of a diamagnetic, upon the lines of force of a magnetic field, which have been described by Faraday.—(*Exp. Researches*, §§ 2807, 2808).]

librium, commencing with A and ending with B, will be the same in the two cases. Hence the resultant force due to the excitation of the dielectric C (or to the imagined distributions of electricity on S and S' which produce it), on points within S or without S', must be such as not to alter the distributions on A and B when the quantity on A is given; and is therefore nothing. Accordingly, let Q be the total force on a point indefinitely near S, and within it; Q' the total force on a point without S', but indefinitely near it. Since the forces on points without S and within S' indefinitely near the former points are, according to the law stated above, $\frac{Q}{k}$ and $\frac{Q'}{k}$, it follows* that the intensities of the imagined distributions on S and S', in the neighbourhood of the points considered, are

$$-\frac{1}{4\pi}\left(Q-\frac{Q}{k}\right) \text{ and } \frac{1}{4\pi}\left(Q'-\frac{Q'}{k}\right).$$

Hence, if U, U' be the potentials at S, S', due to A and B alone, and v the potential at any point P, it follows that the potential at P, due to the polarity of the dielectric, is

$$-\left(1-\frac{1}{k}\right)U+\left(1-\frac{1}{k}\right)U',$$

or
$$-\left(1-\frac{1}{k}\right)v+\left(1-\frac{1}{k}\right)U',$$

or
$$-\left(1-\frac{1}{k}\right)v+\left(1-\frac{1}{k}\right)v, \text{ that is, } 0,$$

according as P is within S, within S' and without S, or without S'. Hence the total potential will be, according to the position of P,

$$v-\left(1-\frac{1}{k}\right)(U-U'),$$

or
$$\frac{v}{k}+\left(1-\frac{1}{k}\right)U',$$

or
$$v.$$

Hence the sole effect of the dielectric C, on the state of A and B, is to diminish the potential in the interior of the former by the quantity

$$\left(1-\frac{1}{k}\right)(U-U').$$

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* See Green's *Essay*, Art. 12; or above, I. § 8.

If the whole space between A and B be occupied by the solid dielectric, the surfaces S and A will coincide, as also, S' and B, and therefore $U = V$, $U' = 0$. Hence the potential in the interior of A will be

$$\frac{V}{k},$$

or the fraction $\frac{1}{k}$ of the potential, with the same charge on A, and with a gaseous dielectric. From this it follows that, when the dielectric is solid, it would require, to produce a given potential in the interior of A, k times the charge which would be necessary to produce the same potential when the dielectric is gaseous, and therefore the body A in a given state, defined by the potential in its interior, produces on the interior surface of B, by induction, through the solid dielectric, a quantity of electricity k times as great as through a gaseous dielectric. On this account Faraday calls the property of a dielectric measured by k, its "specific inductive capacity."

46. In Faraday's experiments an apparatus (which is in fact a Leyden phial, in which any solid or fluid may be substituted for the glass dielectric of an ordinary Leyden phial) is used, corresponding to the case we have been considering, in which A is a conducting sphere (2·33 inches in diameter), and B a concentric spherical shell surrounding it (the distance between the surfaces of A and B being ·62 of an inch). In the shell B there is an aperture into which a shell-lac stem is fixed; a wire, attached to A, passes through the centre of this stem to the outside of the shell, and supports a ball of metal, M, which is thus insulated and connected with A. It may be shown that in such an apparatus the state of the ball A and of the shell B will approximately be not affected by the aperture in the latter, or by the wire supporting M, and that the distribution of electricity on M will be approximately the same as if the wire supporting it and the conductors A and B were removed. Hence the sole relation between A and M will be that the *potentials* in their interiors are the same; and therefore the latter, which is accessible, may be taken as an index of the state of the former.

47. To determine the specific inductive capacity of any dielectric, Faraday uses two apparatus of the kind just described, pre-

cisely equal and similar, in one of which the space between A and B is filled with air, and in the other with the dielectric to be examined. One of these apparatus is charged, and the intensity measured: the balls M, M' in the two are then made to touch and separate again, and the remaining intensity on the first (which is equal to the intensity imparted to the second) is measured. If this be found to differ from half the original intensity, it will follow that the specific inductive capacity of the substance examined differs from that of air, which is unity, and its value may be determined by means of a simple expression from the experimental data. To investigate this, let us first suppose each apparatus to be charged, and let it be required to find the intensity on the balls after they are made to touch, and then removed from mutual influence; and let the dielectrics be any two substances, whose inductive capacities are k, k'. Let ρ, ρ' be the intensities before, and σ the common intensity after contact. Then, denoting by Q, Q' the quantities of electricity constituting the charges before, and q, q' after contact, we shall have, by the principles already developed,

$$\frac{Q}{Q'}=\frac{k\rho}{k'\rho'},\quad \frac{\sigma}{\rho}=\frac{q}{Q},\quad \frac{\sigma}{\rho'}=\frac{q'}{Q'}.$$

Also $$Q+Q'=q+q'.$$

Hence we deduce $$\sigma=\frac{k\rho+k'\rho'}{k+k'}.$$

In the experiment described, one of the dielectrics is air. Hence, to obtain the required formula, we may put $k'=1$, in this equation, and then resolve for k.

Thus we find $$k=\frac{\sigma-\rho'}{\rho-\sigma}.$$

If only one of the apparatus be originally charged, according as it is the first or the second, we shall have

$$k=\frac{\sigma}{\rho-\sigma},$$

or $$k=\frac{\rho'-\sigma}{\sigma}.$$

48. If the substance examined (the dielectric of the first apparatus) be any gas, or air in a different state as to pressure or temperature from the air of the second apparatus, Faraday

always finds the intensity after contact to be half the original intensity, and hence for every gaseous body $k=1$.

49. If the dielectric of the first apparatus be solid, the intensity after contact is found to be greater than half the original intensity when the first, and less than half when the second is the apparatus originally charged. Hence for a solid dielectric, $k>1$. For sulphur Faraday finds the value to be rather more than 2·2; for shell-lac, about 2; and for flint-glass, greater than 1·76.

50. The commonly received ideas of attraction and repulsion exercised at a distance, independently of any intervening medium, are quite consistent with all the phenomena of electrical action which have been here adduced. Thus we may consider the particles of air in the neighbourhood of electrified bodies to be entirely uninfluenced, and therefore to produce no effect in the resultant action on any point: but the particles of a solid non-conductor must be considered as assuming a polarized state when under the influence of free electricity, so as to exercise attractions or repulsions on points at a distance, which, with the action due to the charged surfaces, produce the resultant force at any point. It is, no doubt, possible that such forces at a distance may be discovered to be produced entirely by the action of contiguous particles of some intervening medium, and we have an analogy for this in the case of heat, where certain effects which follow the same laws are undoubtedly propagated from particle to particle. It might also be found that magnetic forces are propagated by means of a second medium, and the force of gravitation by means of a third. We know nothing, however, of the molecular action by which such effects could be produced, and in the present state of physical science it is necessary to admit the known facts in each theory as the foundation of the ultimate laws of action at a distance.

ST. PETER'S COLLEGE,
Nov. 22, 1845.

III. ON THE ELECTRO-STATICAL CAPACITY OF A LEYDEN PHIAL AND OF A TELEGRAPH WIRE INSULATED IN THE AXIS OF A CYLINDRICAL CONDUCTING SHEATH.[*]

[From the *Philosophical Magazine*, 1855, first half-year.]

51. The principles brought forward in the preceding articles On the Uniform Motion of Heat, etc., enable us with great ease to investigate the "capacity"[†] of a Leyden phial with either air, or any liquid or solid dielectric, and of other analogous arrangements, such as the copper wires in gutta-percha tubes under water, with which Faraday has recently performed such remarkable experiments.[‡]

52. Thus, for a Leyden phial, let us suppose a portion S of the surface of a conductor A to be everywhere so near the surface of a conductor A', that the distance between them at any point is a small fraction of the radii of curvature of each surface in the neighbourhood; and let z be the distance between them at a particular position, P. Then, by the analogy with heat, it is clear that if the two surfaces be kept at different electrical potentials, V and V', the potentials at equidistant points in any line across from one to the other will be in arithmetical progression. Hence $\frac{V-V'}{z}$ will be the rate of variation of the potential perpendicularly across in the position P. If, in the first place, the dielectric be air, the electric force in the air

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[*] Communicated as an Additional Note to two papers (I. and II. above) "On the Uniform Motion of Heat in Homogeneous Solid Bodies, and its connexion with the Mathematical Theory of Electricity," and "On the Mathematical Theory of Electricity in Equilibrium;" only not in time to be appended to the reprints of those papers which appeared in the *Philosophical Magazine*, June and July 1854 (1854, I. and II.)

[†] Defined (*Philosophical Magazine*, June 1853) for any conductor (subject or not to the influence of other conductors), as the quantity of electricity which it takes to charge it to unit potential.

[‡] Described in a lecture at the Royal Institution, Jan. 20, 1854, and subsequently published in the *Philosophical Magazine* (1854, I. p. 197).

between the two about the position P will consequently be $\frac{V-V'}{s}$, and therefore the electrical density (according to the theorem proved in the first article) on one surface must be $+\frac{1}{4\pi}\frac{V-V'}{s}$, and on the other $-\frac{1}{4\pi}\frac{V-V'}{s}$. The quantity of electricity in the position P, on an area ds of the surface S, is therefore $\frac{1}{4\pi}\frac{V-V'}{s}ds$, and therefore the whole quantity on S is

$$\frac{V-V'}{4\pi}\int\frac{ds}{s},$$

which is Green's general expression for the electrification of either coating of a Leyden phial. If the thickness of the dielectric be constant and equal to τ, it becomes

$$\frac{V-V'}{4\pi}\frac{S}{\tau}.$$

53. Now if A' be uninsulated, we have $V'=0$; and then, to charge S to the potential V, it takes the quantity $V\times\frac{S}{4\pi\tau}$. Hence the "capacity" of S is

$$\frac{S}{4\pi\tau}.$$

If instead of air there be a solid or liquid dielectric of inductive capacity, k, occupying the space between the two surfaces, the quantity of heat conducted across, in the analogous thermal circumstances, would be k times as great as in the case corresponding to the air dielectric, with the same difference of temperatures; and in the actual electrical arrangement, the quantity of electricity on each of the conducting surfaces would be k times as great as with air for dielectric and the same difference of potentials. The expression for the capacity of an actual Leyden phial is therefore

$$\frac{kS}{4\pi\tau},$$

k being the inductive capacity of the solid non-conductor of which it is formed, τ its thickness, and S the area of it which is coated on each side.

54. To investigate the capacity of a copper wire in the circumstances experimented on by Faraday, let us first consider the analogous circumstances regarding the conduction of heat; that is, let us consider the conduction of heat that would take place

across the gutta-percha, if the copper wire in its interior were kept continually at a temperature a little above that of the water which surrounds it. Here the quantity of heat flowing outwards from any length of the copper wire, the quantities flowing across different surfaces surrounding it in the gutta-percha, and the quantity flowing into the water from the same length of gutta-percha tube, in the same time, must be equal. But the areas of the same length of different cylindrical surfaces are proportional to their radii, and therefore the flow of heat across equal areas of different cylindrical surfaces in the gutta-percha, coaxial with the wire, must be inversely as their radii. Hence, in the corresponding electrical problem, with air as the dielectric instead of gutta-percha, if R denote the resultant electrical force at any point P in the air between an insulated, electrified, infinitely long cylindrical conductor, and an uninsulated, coaxial, hollow cylindrical conductor surrounding it, and if x be the distance of P from the axis, we have

$$R=\frac{A}{x},$$

where A denotes a constant. But if v be the potential at P; by the definition of "potential" we have

$$\frac{dv}{dx}=-R.$$

Hence

$$\frac{dv}{dx}=-\frac{A}{x};$$

and, by integration,

$$v=-A\log x+C.$$

Assigning the constants A and C so that the potential may have the value V at the surface of the wire, and may vanish at the hollow conducting surface round it, if r and r' denote the radii of these cylinders respectively, we have

$$v=V\frac{\log\frac{r'}{x}}{\log\frac{r'}{r}},$$

and

$$-\frac{dv}{dx}=R=\frac{V}{\log\frac{r'}{r}}\frac{1}{x}.$$

55. Taking $x=r$, we find by this the electric force in the air

infinitely near the inner electrified conductor; and dividing the value found, by 4π (according to the general theorem), we have

$$\frac{1}{4\pi}\frac{V}{r\log\frac{r'}{r}}$$

for the electrical density on the surface of the conductor. Multiplying this by $2\pi rl$, the area of a length l of the surface, we find

$$\tfrac{1}{2}\frac{Vl}{\log\frac{r'}{r}}$$

for the whole quantity of electricity on that length. Hence, if k be the specific inductive capacity of gutta-percha, the electricity resting on a length l of the wire in the actual circumstances will amount to

$$\tfrac{1}{2}\frac{kl}{\log\frac{r'}{r}}V.$$

Or if S denote the surface of the wire, we have, for the quantity of electricity which it holds,

$$V.\frac{kS}{4\pi r\log\frac{r'}{r}};$$

and therefore its capacity is the same as that of a Leyden phial with an equal area of coated glass of thickness equal to $\frac{I}{k}r\log\frac{r'}{r}$, if I denote the specific inductive capacity of the glass.

56. In the case experimented on by Mr. Faraday, the diameter of the wire was $\frac{1}{16}$th of an inch, and the exterior diameter of the gutta-percha covering was about four times as great. Hence the thickness of the equivalent Leyden phial must have been

$$\frac{I}{k}\cdot\frac{1}{32}\log_e 4 = \frac{I}{k}\cdot\frac{1}{23\cdot08}.$$

As the surface of the wire amounted to 8300 square feet, we may infer that if the gutta-percha had only the same inductive capacity as glass (and it probably has a little greater), the insulated wire, when the outer surface of the gutta-percha was uninsulated, would have had an electrical capacity equal to that of an ordinary Leyden battery of 8300 square feet of coated glass $\frac{1}{23}$rd of an inch thick.

INVERCLOY, ARRAN, *June* 1854.

IV. ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

II.—A STATEMENT OF THE PRINCIPLES ON WHICH THE MATHEMATICAL THEORY OF ELECTRICITY IS FOUNDED.

[*Cambridge and Dublin Mathematical Journal*, March 1848.]

57. This paper may be regarded as introductory to some others which will follow, containing various investigations in the Theory of Electricity. The fundamental mathematical principles of the phenomena of Electricity in Equilibrium are stated and explained in as concise a manner as seems consistent with clearness. To avoid lengthening the paper and unnecessarily distracting the attention of the reader, no details are given with reference to the experiments which have been, or which might be, made for establishing the various propositions asserted; and, for the same reasons, scarcely any allusion is made to the history of the subject. With regard to the nature of the evidence on which the mathematical theory of electricity rests, the reader is referred to the preceding paper "On the Elementary Laws of Statical Electricity," where, besides some general explanations on the subject, the works containing accounts of the actual experimental researches of principal importance are indicated. That paper is marked as the first of a series which it was my intention to publish in this *Journal*, and of which the second now appears. In this series it will not be attempted to adhere to a systematic course of investigations such as might constitute a complete treatise on the subject; and my only reason for publishing this introductory article is for the sake of reference in other papers, there being no published work in which the principles are stated in a sufficiently concise and correct form, independently of any hypothesis, to be altogether satisfactory in the present state of science.

The Two Kinds of Electricity.

58. If a piece of glass and a piece of resin are rubbed together and then separated, it is found that they attract one another

mutually. The term *electricity*[*] has been applied to the agency developed in this operation; the excitation of the bodies, to which the attractive force is due, is called *electrical*, and the bodies so excited are said to be *electrified*, or to be *charged with electricity*.

If second pieces of glass and resin be rubbed together and then separated, and placed in the neighbourhood of the first pair of electrified bodies, it may be observed—

(1) *That the two pieces of glass repel one another.*
(2) *That each piece of glass attracts each piece of resin.*
(3) *That the two pieces of resin repel one another.*

Hence it is inferred that the two pieces of glass possess electrical properties which differ in their characteristics from those of the resin; and the two kinds of electricity thus indicated are called vitreous and resinous, after the substances on which they are developed. Bodies may in various ways be made electric; but the characteristics presented are always those of either *vitreous electricity* or *resinous electricity*.

59. An electrified body exerts no force, whether of attraction or of repulsion, upon any non-electric matter. When in any case bodies not previously electrified are observed to be attracted, or urged in any direction, by an electrical mass, it is because the bodies have become *electrically excited by influence*.

60. If a small piece of glass and a small piece of resin, which have been electrified by mutual friction, be placed successively in the same position in the neighbourhood of an electrified body, they will be acted upon by equal forces, in the same line, but in contrary directions. Hence the two bodies are said to be *equally charged* with the two kinds of electricity respectively.

Electrical Quantity.

61. The force between two electrified bodies depends, *cæteris paribus*, on the amounts of their charges, or on the quantities of electricity which they possess.

If a small piece of glass and a small piece of resin be electrified by mutual friction to such an extent that, when separated and placed at a unit of distance, they attract one another with a unit of force, the quantity of electricity possessed by the former is said to be unity; the latter possesses what may be called a unit of resinous electricity.

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[*] From ἤλεκτρον, amber, on account of such phenomena having been first observed with amber as one of the substances rubbed together.

If m bodies, each possessing a unit of vitreous electricity, be incorporated together, the single body thus composed is charged with m units of the same kind of electricity: It is said to possess a quantity of electricity equal to m, or its *electrical mass* is m. A similar definition is applicable with reference to the measurement of resinous electricity.

62. If two bodies possessing equal quantities of vitreous and resinous electricity be incorporated, the single body thus composed will be found either to be non-electric, or to be in such a state that, without the removal of any electricity of either kind from it, it may, merely by an alteration in the distribution of what it already possesses, be deprived of all electrical symptoms. Thus it appears that a body either vitreously or resinously electrified, may be deprived of its charge merely by supplying it with an equal quantity of the other kind of electricity.

In consequence of this fact, we may establish a complete system of algebraic notation with reference to electrical quantity, whether of vitreous or resinous electricity, by adopting as universal the law that the total quantity of electricity possessed by two bodies, or the quantity possessed by one body made up of two, is equal to the sum of the quantities with which they are separately charged. Thus let m be the quantity of electricity with which a vitreously electrified body is charged, and let m' be the quantity contained by a body equally charged with resinous electricity. We must have

$$m + m' = 0,$$

and therefore m' is equal to $-m$. Now it is usual to regard vitreous electricity as positive; and we must therefore regard the other kind as negative; so that a body possessing m units of resinous electricity is to be considered as charged with a quantity $-m$ of electricity.

The Superposition of Electrical Forces.

63. If a body, electrified in a given invariable manner, be placed in the neighbourhood of any number of electrified bodies, it will experience a force which is the resultant of the forces that would be separately exerted upon it by the different bodies

if they were placed in succession in the positions which they actually occupy, without any alteration in their electrical conditions.

This law is true even if any number of the bodies considered be merely different parts of one continuous mass.

COR. 1. The total mechanical action between two electrified bodies, whether parts of one continuous mass or isolated bodies, is the resultant of the forces due to the mutual actions between all parts of either body and all parts of the other, if we conceive the two bodies to be arbitrarily divided each into parts in any manner whatever.

COR. 2. We may, in any electrical problem, imagine the charge possessed by a body to be divided into two or more parts, each distributed arbitrarily with the sole condition that the sum of the quantities of electricity in any very small space of the body due to the different distributions shall be equal to the given quantity of electricity in that space, according to the actual distribution of electricity in the body; and we may consider the force actually exerted upon any other electrified body as equivalent to the resultant of the forces due to these partial distributions.

The Law of Force between Electrified Bodies.

64. The force between two small electrified bodies varies inversely as the square of the distance between them.

COR. If two small bodies be charged respectively with quantities m and m' of electricity, they will mutually repel with a force equal to

$$\frac{mm'}{\Delta^2};$$

(an action which will be really attractive when m and m' have unlike signs, as would be the case were the bodies dissimilarly electrified). For two units, placed at a distance unity, repel with a force equal to unity, and therefore if placed at a distance Δ, they will repel with a force $\frac{1}{\Delta^2}$; and the expression for the repulsion between m units and m' units is deduced from this, according to the principle of the superposition of forces, by multiplying by mm'.

Definition of the Resultant Electrical Force at a Point.

65. Let a unit of negative electricity be conceived to be concentrated at a point P in the neighbourhood of an electrified body or group of bodies, without producing any alteration in the previously existing electrical distribution. The force exerted upon this electrical point is what we shall throughout understand *as the resultant force at* P due to the electricity of the body or bodies considered.

COR. If R be *the resultant force at* P in any case, then the force actually exerted upon an electrical mass m, concentrated at P, will be equal to $-mR$.

Electrical Equilibrium.

66. When a body held at rest is electrified, and when, being either subject to electrical action from other bodies, or entirely isolated, the distribution of its charge remains permanently unaltered, the electricity upon it is said to be in equilibrium.

Electrical equilibrium may be disturbed in various ways. Thus if a body charged with electricity in equilibrium be touched, or even approached by another electrified body, the equilibrium may be broken, and can only be restored after a different distribution has been effected, by a motion of electricity through the body or along its surface: or if a body be initially electrified in any arbitrary manner, whether by friction or otherwise, it may be that, as soon as the exciting cause is removed, the electricity will either gradually become altered from its initial distribution, by moving slowly through the body, or will suddenly assume a certain definite distribution.

The laws which regulate the distribution of electricity in equilibrium on bodies in various circumstances have been the subject of most important experimental researches; and having been established with perfect precision by Coulomb, and placed beyond all doubt by verifications afforded in subsequent experiments, they constitute the foundation of an extremely interesting branch of the Mathematical Theory of Electricity. In connexion with these laws, and before stating them, it will be convenient to explain the nature of the distinction which is drawn between the two great classes of bodies in nature, called Conductors of Electricity, and Non-Conductors of Electricity.

Non-Conductors of Electricity.

67. A body which affords such a resistance to the transmission of electricity through it, or along its surface, that, if it be once electrified in any way, it retains permanently, without any change of distribution, the charge which it has received, is called a Non-Conductor of Electricity.

No body exists in nature which fulfils strictly the terms of this definition; but glass and resin, besides many other substances, are such that they may, within certain limits and subject to certain restrictions, be considered as non-conductors.

Conductors of Electricity.

68. A very extensive class of bodies in nature, including all the metals, many liquids, etc., are found to possess the property that, in all conceivable circumstances of electrical excitation, the resultant force at any point within their substance vanishes. Such bodies are called Conductors of Electricity, since they are destitute of the property, possessed by non-conductors, of retaining permanently, by a resistance to every change, any distribution of electricity arbitrarily imposed; the only kind of distribution which can exist unchanged for an instant being such as satisfies the condition that the resultant force must vanish in the interior.

It is found by experiment that the electricity of a charged conductor rests entirely on its surface, and that the electrical circumstances are not at all affected by the nature of the interior, but depend solely upon the form of the external conducting surface. Thus the electrical properties of a solid conductor, of a hollow conducting shell, or of a non-conductor enclosed in an envelop, however thin (the finest gold leaf, for instance), are identical, provided the external forms be the same. A hollow conductor never shows symptoms of electricity on its interior surface, unless an electrified body be insulated within it; in which case the interior surface will become electrified by *influence*, or by *induction*, in such a way as to make the total resultant force at any point in the conducting matter vanish, by balancing, for any such point, the force due to the electricity of the insulated body.

It has been frequently assumed that electricity penetrates to a finite depth below the surface of conductors; and, in accordance with certain hypothetical ideas regarding the nature of electricity, the "thickness of the stratum" at different points of the surface of a conductor has been considered as a suitable term with reference to the varying or uniform distribution of electricity over the body. All the conclusion with reference to this delicate subject which can as yet be drawn from experiment, is that the "thickness," if it exist at all, must be less than that of the finest gold leaf; and in the present state of science we must regard it as immeasurably small. It may be conceived that the actual thickness of the excited stratum at the surface of an electrified conductor is of the same order as the space through which the physical properties of the pervading matter change continuously from those of the solids to those which characterize the surrounding air.

Electrical Density at any Point of a Charged Surface.

69. In this, and in all the papers which will follow, instead of the expression "the thickness of the stratum," Coulomb's far more philosophical term, *Electrical Density*, will be employed with reference to the distribution of electricity on the surface of a body; a term which is to be understood strictly in accordance to the following definitions, without involving even the idea of a hypothesis regarding the nature of electricity. The electrical density of a uniformly charged surface is the quantity of electricity distributed over a unit of surface.

The electrical density at any point of a surface, whether the distribution be uniform or not, is the quotient obtained by dividing the quantity of electricity distributed over an infinitely small element at this point, by the area of the element.

Exclusion of all Non-Conductors except Air.

70. In the present paper, and in some others which will follow, no bodies will be considered except conductors; and the air surrounding them, which will be considered as offering a resistance to the transference of electricity between two detached conductors, but as otherwise destitute of electrical properties. A full development of the mathematical theory, of the internal electrical polarization of solid or liquid non-con-

ductors, subject to the influence of electrified bodies, discovered by Faraday (in his Experimental Researches on the specific inductive capacities of non-conducting media), must be reserved for a later communication.*

Insulated Conductors.

71. A conductor separated from the ground, and touched only by air, is said to be insulated. Insulation may be practically effected by means of solid props of matter, such as glass, shell-lac, or gutta percha;† and if the props be sufficiently thin, it is found that their presence does not in any way alter or affect the electrical circumstances, and that their *resisting* power, as non-conductors of electricity, prevents any alteration in the quantity of electricity possessed by the insulated body; so that however the distribution may be affected by the influence of surrounding bodies, it is only by a temporary breaking of the insulation that the absolute charge can be increased or diminished.

If an insulated uncharged conductor be placed in the neighbourhood of bodies charged with electricity, it will become "electrified by influence," in such a manner that its resultant electrical force at every internal point shall counterbalance the force due to the exterior charged bodies: but, in accordance with what has been stated in the preceding paragraph, the total quantity of electricity will remain equal to nothing; that is to say, the two kinds of electricity produced upon it by influence will be equal to one another in amount.

Recapitulation of the Fundamental Laws.

72. The laws of electricity in equilibrium in relation with conductors may—if we tacitly take into account such principles

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* The results of this Theory were explained briefly in a paper entitled "Note sur les Lois Elémentaires de l'Electricité Statique" (published, in 1845, in *Liouville's Journal*), and more fully in the first paper of the present series, on the "Mathematical Theory of Electricity" (II. above). A similar view of this subject has been taken by Mossotti, whose investigations are published in a paper entitled "*Discussione Analitica sull' Influenza che l' Azione di un Mezzo Dielettrico ha sulla Distribuzione dell' Elettricità alla Superficie di piu Corpi Elettrici Disseminati in Esso.* (Vol. XXIV. of the *Memorie della Società Italiana delle Scienze Residente in Modena*, dated 1846.)

† It has been recently discovered by Faraday that gutta percha is one of the best insulators among known substances—(*Phil. Mag.*, March 1848.)

as the superposition of electrical forces, and the invariableness of the quantity of electricity on a body, except by addition or subtraction (in the extended algebraic sense of these terms)—be considered as fully expressed in the three following propositions:—

I. The repulsion between two electrical points is inversely proportional to the square of their distance.

II. Electricity resides at the boundary of a charged conductor.

III. The resultant force at any point in the substance of a conductor, due to all existing electrified bodies, vanishes.

It has been proved by Green that the second of these laws is a mathematical consequence of the first and third; and it has been demonstrated by La Place* that the first law may be inferred from the truth, in a certain particular case, of the second and third. The three laws were, however, first announced by Coulomb, as the result of his experimental researches on the subject.

Objects of the Mathematical Theory of Electricity.

73. The varied problems which occur in the mathematical theory of electricity in equilibrium may be divided into the two great classes of Synthetical and Analytical investigations. In problems of the former class, the object is in each case the determination either of a resultant force or of an aggregate electrical mass, according to special data regarding distributions of electricity: in the latter class, inverse problems, such as the determination of the electrical density at each point of the surface of a conductor in any circumstances, according to the laws stated above, are the objects proposed.

It has been proved (by Green and Gauss) that there is a determinate unique solution of every actual analytical problem of the Theory of Electricity in relation with conductors. The demonstration of this with reference to the complete Theory of Electricity (including the action of solid non-conducting media discovered by Faraday), as well as with reference to the Theories of Heat, Magnetism, and Hydrodynamics, may be deduced from two theorems proved in the Cambridge and Dublin Mathematical Journal for 1847, "Regarding the Solution of certain Partial

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* [Originally by Cavendish, as I learned after the first publication of this paper. See footnote of March 1854 on § 24 above.]

Differential Equations" (XIII. below, or Thomson and Tait's *Natural Philosophy*, App. A.)

The full investigation of any actual case of electrical equilibrium will generally involve both analytical and synthetical problems; as it may be desirable, besides determining the distribution, to find the resulting electrical force at points not in the interior of any conductor, or to find the total mechanical action due to the attractions or repulsions of the elements of two conductors, or of two portions of one conductor; and besides, it is frequently interesting to verify synthetically the solutions obtained for analytical problems.

Actual Progress in the Mathematical Theory of Electricity.

74. In Poisson's valuable memoirs on this subject, the distribution of electricity on two electrified spheres, uninfluenced by other electric matter, is considered; a complete solution of the analytical problem is arrived at; and various special cases of interest are examined in detail with great rigor. In a very elaborate memoir by Plana,[*] the solution given by Poisson is worked out much more fully, the excessive mathematical difficulties in the way of many actual numerical applications of interest being such as to render a work of this kind extremely important.

The distribution of electricity on an ellipsoid (including the extreme cases of elliptic and circular discs, and of a straight rod), and the results of consequent *synthetical* investigations are well known.

The analytical problem regarding an ellipsoid subject to the influence of given electrical masses, has been solved by M. Liouville, by the aid of a very refined mathematical method suggested by some investigations of M. Lamé with reference to corresponding problems in the Theory of Heat.

Green's Essay on Electricity and his other papers on allied subjects contain, besides the solution of several special problems of interest, most valuable discoveries with reference to the general Theory of Attraction, and open the way to much more extended investigations in the Theory of Electricity than any that have yet been published.

GLASGOW COLLEGE, *March* 4, 1848.

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[*] *Turin Academy of Sciences*, tome vii. Série II. published separately in a quarto volume of 323 pages: Turin, 1845.

V. ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

III.—GEOMETRICAL INVESTIGATIONS WITH REFERENCE TO THE DISTRIBUTION OF ELECTRICITY ON SPHERICAL CONDUCTORS.*

[*Cambridge and Dublin Mathematical Journal*, March, May, and Nov. 1848, Nov. 1849, Feb. 1850.]

75. There is no branch of physical science which affords a surer foundation, or more definite objects for the application of mathematical reasoning, than the theory of electricity. The small amount of attention which this most attractive subject has obtained is no doubt owing to the extreme difficulty of the analysis by which even a very limited progress has as yet been made; and no other circumstance could have totally excluded from an elementary course of reading, a subject which, besides its great physical importance, abounds so much in beautiful illustrations of ordinary mechanical principles. This character of difficulty and impracticability is not however inseparable from the mathematical theory of electricity: by very elementary geometrical investigations we may arrive at the solution

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* The investigations given in this paper (§§ 75-127) form the subject of the first part of a series of lectures on the Mathematical Theory of Electricity given in the University of Glasgow during the present session [1847-8]. They are adaptations of certain methods of proof which first occurred to me as applications of *the principle of electrical images*, made with a view to investigating the solutions of various problems regarding spherical conductors, without the explicit use of the differential or integral calculus. The spirit, if not the notation, of the *differential* calculus must enter into any investigations with reference to Green's theory of the potential, and therefore a more extended view of the subject is reserved for a second part of the course of lectures. A complete exposition of the *principle of electrical images* (of which a short account was read at the late meeting of the British Association at Oxford) has not yet been published; but an outline of it was communicated by me to M. Liouville in three letters, of which extracts are published in the *Journal de Mathématiques* (1845 & 1847, vols. x., xii.) [See XIV. below.] A full and elegant exposition of the method indicated, together with some highly interesting applications to problems in geometry not contemplated by me, are given by M. Liouville himself, in an article written with reference to those letters, and published along with the last of them. I cannot neglect the present opportunity of expressing my thanks for the honour which has thus been conferred upon me by so distinguished a mathematician, as well as for the kind manner in which he received those communications, imperfect as they were, and for the favourable mention made of them in his own valuable memoir.

of a great variety of interesting problems with reference to the distribution of electricity on spherical conductors, including Poisson's celebrated problem of the two spheres, and others which might at first sight be regarded as presenting difficulties of a far higher order. The object of the following paper is to present, in as simple a form as possible, some investigations of this kind. The methods followed, being for the most part *synthetical*, were suggested by a knowledge of results founded on a less restricted view of the theory of electricity; and it must not be considered either that they constitute the best or the easiest way of advancing towards a *complete* knowledge of the subject, or that they would be suitable as instruments of research in endeavouring to arrive at the solutions of new problems.

Insulated Conducting Sphere subject to no External Influence.

76. We may commence with the simplest possible case, that of a spherical conductor, charged with electricity and insulated in a position removed from all other bodies which could influence the distribution of its charge. In this, as in the other cases which will be considered, the various problems, of the analytical and synthetical classes, alluded to in a previous paper (IV. § 73) will be successively subjects of investigation. Thus let us first determine the density at any point of the surface, and then, after verifying the result by showing that the laws (§ 72) are satisfied, let us investigate the resultant force at an external point.

Determination of the Distribution.

77. Let a be the radius of the sphere, and E the amount of the charge.

According to Law II., the whole charge will reside on the surface, and, on account of the symmetry, it must be uniformly distributed. Hence, if ρ be the required density at any point, we have,

$$\rho = \frac{E}{4\pi a^2}.$$

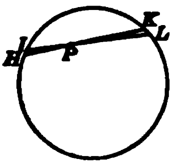
Verification of Law III.

78. The well-known theorem, that the resultant force due to a uniform spherical shell vanishes for any interior point, constitutes the verification required in this case. This theorem was

first given by Newton, and is to be found in the *Principia*; but as his demonstration is the foundation of every synthetical investigation which will be given in this paper, it may not be superfluous to insert it here; and accordingly the passage of the *Principia* in which it occurs, translated literally, is given here.

Newton, First Book, Twelfth Section, Prop. LXX. Theorem XXX.

If the different points of a spherical surface attract equally with forces varying inversely as the squares of the distances, a particle placed within the surface is not attracted in any direction.



Let $HIKL$ be the spherical surface, and P the particle within it. Let two lines HK, IL, intercepting very small arcs HI, KL, be drawn through P; then on account of the similar triangles HPI, KPL (Cor. 3, Lemma VII. Newton), those arcs will be proportional to the distances HP, LP; and any small elements of the spherical surface at HI and KL, each bounded all round by straight lines passing through P [and very nearly coinciding with HK], will be in the duplicate ratio of those lines. Hence the forces exercised by the matter of these elements on the particle P are equal; for they are as the quantities of matter directly, and the squares of the distances, inversely; and these two ratios compounded give that of equality. The attractions therefore, being equal and opposite, destroy one another: and a similar proof shows that all the attractions due to the whole spherical surface are destroyed by contrary attractions. Hence the particle P is not urged in any direction by these attractions. Q. E. D.

Digression on the Division of Surfaces into Elements.

79. The division of a spherical surface into infinitely small elements will frequently occur in the investigations which follow: and Newton's method, described in the preceding demonstration, in which the division is effected in such a manner that all the parts may be taken together in *pairs of opposite elements with reference to an internal point*; besides other

methods deduced from it, suitable to the special problems to be examined; will be repeatedly employed. The present digression, in which some definitions and elementary geometrical propositions regarding this subject are laid down, will simplify the subsequent demonstrations, both by enabling us, through the use of convenient terms, to avoid circumlocution, and by affording us convenient means of reference for elementary principles, regarding which repeated explanations might otherwise be necessary.

Explanations and Definitions regarding Cones.

80. If a straight line which constantly passes through a fixed point be moved in any manner, it is said to describe, or generate, a *conical surface* of which the fixed point is the vertex.

If the generating line be carried from a given position continuously through any series of positions, no two of which coincide, till it is brought back to the first, the entire line on the two sides of the fixed point will generate a complete conical surface, consisting of two sheets, which are called *vertical or opposite cones*. Thus the elements HI and KL, described in Newton's demonstration given above, may be considered as being cut from the spherical surface by two *opposite cones* having P for their common vertex.

The Solid Angle of a Cone, or of a complete Conical Surface.

81. If any number of spheres be described from the vertex of a cone as centre, the segments cut from the concentric spherical surfaces will be similar, and their areas will be as the squares of the radii. The quotient obtained by dividing the area of one of these segments by the square of the radius of the spherical surface from which it is cut, is taken as the measure of the *solid angle of the cone*. The segments of the same spherical surfaces made by the opposite cone, are respectively equal and similar to the former. Hence the solid angles of two vertical or opposite cones are equal: either may be taken as the solid angle of the complete conical surface, of which the opposite cones are the two sheets.

Sum of all the Solid Angles round a Point = 4π.

82. Since the area of a spherical surface is equal to the square of its radius multiplied by 4π, it follows that the sum of the solid angles of all the distinct cones which can be described with a given point as vertex, is equal to 4π.

Sum of the Solid Angles of all the complete Conical Surfaces = 2π.

83. The solid angles of vertical or opposite cones being equal, we may infer from what precedes that the sum of the solid angles of all the complete conical surfaces which can be described without mutual intersection, with a given point as vertex, is equal to 2π.

Solid Angle subtended at a Point by a Terminated Surface.

84. The solid angle subtended at a point by a superficial area of any kind, is the solid angle of the cone generated by a straight line passing through the point, and carried entirely round the boundary of the area.

Orthogonal and Oblique Sections of a Small Cone.

85. A very small cone, that is, a cone such that any two positions of the generating line contain but a very small angle, is said to be cut at right angles, or orthogonally, by a spherical surface described from its vertex as centre, or by any surface, whether plane or curved, which touches the spherical surface at the part where the cone is cut by it.

A very small cone is said to be cut obliquely, when the section is inclined at any finite angle to an orthogonal section; and this angle of inclination is called the *obliquity of the section.*

The area of an orthogonal section of a very small cone is equal to the area of an oblique section in the same position, multiplied by the cosine of the obliquity.

Hence the area of an oblique section of a small cone is equal to the quotient obtained by dividing the product of the square of its distance from the vertex, into the solid angle, by the cosine of the obliquity.

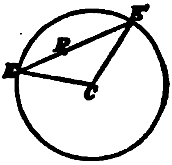
Area of the Segment cut from a Spherical Surface by a Small Cone.

86. Let E denote the area of a very small element of a

spherical surface at the point E (that is to say, an element every part of which is very near the point E), let ω denote the solid angle subtended by E at any point P, and let PE, produced if necessary, meet the surface again in E': then, a denoting the radius of the spherical surface, we have

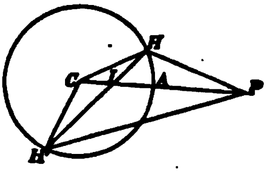
$$E = \frac{2a.\omega.PE^2}{EE'}.$$

For, the obliquity of the element E, considered as a section of the cone of which P is the vertex and the element E, a section; being the angle between the given spherical surface and another described from P as centre, with PE as radius; is equal to the angle between the radii, EP and EC, of the two spheres. Hence, by considering the isosceles triangle ECE', we find that the cosine of the obliquity is equal to $\frac{\frac{1}{2}EE'}{EC}$, or to $\frac{EE'}{2a}$, and we arrive at the preceding expression for E.



87. *Theorem.*[*] The attraction of a uniform spherical surface on an external point is the same as if the whole mass were collected at the centre.

Let P be the external point, C the centre of the sphere, and CAP a straight line cutting the spherical surface in A. Take I in CP, so that CP, CA, CI may be continual proportionals, and let the whole spherical surface be divided into *pairs of opposite elements with reference to the point I.*



Let H and H' denote the magnitudes of a pair of such elements, situated respectively at the extremities of a chord HH'; and let ω denote the magnitude of the solid angle subtended by either of these elements at the point I.

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[*] This theorem, which is more comprehensive than that of Newton in his first proposition regarding attraction on an external point (Prop. LXXI.), is fully established as a corollary to a subsequent proposition (Prop. LXXIII. Cor. 2). If we had considered the proportion of the forces exerted upon two external points at different distances, instead of, as in the text, investigating the absolute force on one point, and if besides we had taken together all the pairs of elements which would constitute two narrow annular portions of the surface, in planes perpendicular to PC, the theorem and its demonstration would have coincided precisely with Prop. LXXI. of the *Principia.*

We have (§ 85),

$$H = \frac{\omega . IH^2}{\cos CHI}, \text{ and } H' = \frac{\omega . IH'^2}{\cos CH'I}.$$

Hence, if ρ denote the density of the surface (§ 69), the attractions of the two elements H and H' on P are respectively

$$\rho \frac{\omega}{\cos CHI} . \frac{IH^2}{PH^2}, \text{ and } \rho \frac{\omega}{\cos CH'I} . \frac{IH'^2}{PH'^2}.$$

Now the two triangles PCH, HCI have a common angle at C, and, since $PC : CH :: CH : CI$, the sides about this angle are proportional. Hence the triangles are similar; so that the angles CPH and CHI are equal, and

$$\frac{IH}{HP} = \frac{CH}{CP} = \frac{a}{CP}.^{*}$$

In the same way it may be proved, by considering the triangles PCH', $H'CI$, that the angles CPH' and $CH'I$ are equal, and that

$$\frac{IH'}{H'P} = \frac{CH'}{CP} = \frac{a}{CP}.$$

Hence the expressions for the attractions of the elements H and H' on P become

$$\rho \frac{\omega}{\cos CHI} . \frac{a^2}{CP^2}, \text{ and } \rho \frac{\omega}{\cos CH'I} . \frac{a^2}{CP^2},$$

which are equal, since the triangle HCH' is isosceles; and, for the same reason, the angles CPH, CPH', which have been proved to be respectively equal to the angles CHI, $CH'I$, are equal. We infer that the resultant of the forces due to the two elements is in the direction PC, and is equal to

$$2\omega . \rho . \frac{a^2}{CP^2}.$$

To find the total force on P, we must take the sum of all

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[*] From this we infer that the ratio of IH to HP is constant, whatever be the position of H on the spherical surface, a well-known proposition.—(Thomson's *Euclid*, vi. Prop. G.)

the forces along PC due to the pairs of opposite elements; and, since the multiplier of ω is the same for each pair, we must add all the values of ω, and we therefore obtain (§ 83), for the required resultant,

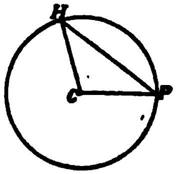
$$\frac{4\pi\rho a^2}{CP^2}.$$

The numerator of this expression; being the product of the density, into the area of the spherical surface; is equal to the mass of the entire charge; and therefore the force on P is the same as if the whole mass were collected at C. Q. E. D.

COR. The force on an external point, infinitely near the surface, is equal to $4\pi\rho$, and is in the direction of a normal at the point. The force on an internal point, however near the surface, is, by a preceding proposition, equal to nothing.

Repulsion on an element of the Electrified Surface.

88. Let σ be the area of an infinitely small element of the surface at any point P, and at any other point H of the surface let a small element subtending a solid angle ω, at P, be taken. The area of this element will be equal to



$$\frac{\omega . PH^2}{\cos CHP},$$

and therefore the repulsion along HP, which it exerts on the element σ at P, will be equal to

$$\frac{\rho\omega . \rho\sigma}{\cos CHP}, \text{ or } \frac{\omega}{\cos CHP}\rho^2\sigma.$$

Now the total repulsion on the element at P is in the direction CP; the component in this direction of the repulsion due to the element H, is

$$\omega . \rho^2\sigma;$$

and, since all the cones corresponding to the different elements of the spherical surface lie on the same side of the tangent plane at P, we deduce, for the resultant repulsion on the element σ,

$$2\pi\rho^2\sigma.$$

From the corollary to the preceding proposition, it follows that this repulsion is half the force which would be exerted on an external point, possessing the same quantity of electricity as the element σ, and placed infinitely near the surface.

GLASGOW COLLEGE, *March* 14, 1848.

INSULATED SPHERE SUBJECTED TO THE INFLUENCE OF AN ELECTRICAL POINT—(§§ 89-95).

89. A conducting sphere placed in the neighbourhood of an electrified body must necessarily become itself electric, even if it were previously uncharged; since (Law III.) the entire resultant force at any point within it must vanish, and consequently there must be a distribution of electricity on its surface which will for internal points balance the force resulting from the external electrified body. If the sphere, being insulated, be previously charged with a given quantity of electricity, the whole amount will (§ 71) remain unaltered by the electrical influence, but its distribution cannot be uniform, since in that case, it would exert no force on an internal point, and there would remain the unbalanced resultant due to the external body. In what follows, it will be proved that the conditions are satisfied by a certain assumed distribution of electricity in each instance; but the proposition that no other distribution can satisfy the conditions, which is merely a case of a general theorem referred to above (§ 73), will not be specially demonstrated with reference to the particular problems; although we shall have to assume its truth when a certain distribution which is proved synthetically to satisfy the conditions is asserted to be the unique solution of the problem.

Attraction of a Spherical Surface of which the density varies inversely as the cube of the distance from a given point.

90. Let us first consider the case in which the given point S and the attracted point P are separated by the spherical surface. The two figures represent the varieties of this case in which the point S being without the sphere, P is within; and, S being within, the attracted point is external. The same demonstration is applicable literally with reference to the two

figures; but, for avoiding the consideration of negative quantities, some of the expressions may be conveniently modified to suit the second figure. In such instances the two expressions are given in a double line, the upper being that which is most convenient for the first figure, and the lower for the second.

Let the radius of the sphere be denoted by a, and let f be the distance of S from C, the centre of the sphere (not represented in the figures).

Join SP and take T in this line (or its continuation) so that

$$\left.\begin{array}{ll}\text{(fig. 1.)} & SP.ST=f^2-a^2 \\ \text{(fig. 2.)} & SP.TS=a^2-f^2\end{array}\right\} \qquad (1).[*]$$

Through T draw any line cutting the spherical surface at K, K'. Join SK, SK', and let the lines so drawn cut the spherical surface again in EE'.

Let the whole spherical surface be divided into pairs of opposite elements with reference to the point T. Let K and K' be a pair of such elements situated at the extremities of the chord KK', and subtending the solid angle ω at the point T; and let elements E and E' be taken subtending at S the same solid angles respectively as the elements K and K'. By this means we may divide the whole spherical surface into pairs of conjugate elements, E, E', since it is easily seen that when we have taken every pair of elements, K, K', the whole surface will have been exhausted, without repetition, by the deduced ele-

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[*] If, in geometrical investigations in which diagrams are referred to, the distinction of *positive* and *negative* quantities be observed, the order of the letters expressing a straight line will determine the algebraic sign of the *quantity* denoted: thus we should have, universally, if A, B be the extremities of a straight line, $AB=-BA$, each member of this equation being positive or negative according to the conventional direction in which positive quantities are estimated. In the present instance, lengths measured along the line SP in the direction from S towards P, or in corresponding directions in the continuation of this line on either side, are, in both figures, considered as positive. Hence, in the first figure ST will be positive; but when f is less than a, ST must be negative on account of the equation $SP.ST=f^2-a^2$. Hence the second figure represents this case; and, if we wish to express the circumstances without the use of negative quantities, we must change the signs of both members of the equation, and substitute for the positive quantity $-ST$ its equivalent TS, so that we have $SP.TS=a^2-f^2$, as the most convenient form of the expression, when reference is made to the second figure. See above (Symbolical Geometry, § 4), in volume of the *Cambridge and Dublin Mathematical Journal* for 1848, where the principles of interpretation of the sign $-$ in geometry are laid down by Sir William R. Hamilton [or Tait's *Quaternions*, § 20, 1868].

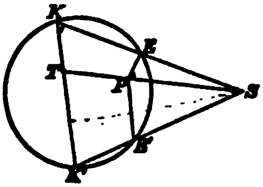


FIG. 1.

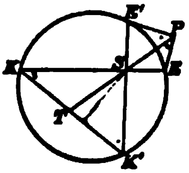


FIG. 2.

ments, E, E'. Hence the attraction on P will be the final resultant of the attractions of all the pairs of elements, E E'.

Now if ρ be the electrical density at E, and if F denote the attraction of the element E on P, we have

$$F=\frac{\rho . E}{EP^2}.$$

According to the given law of density we shall have

$$\rho=\frac{\lambda}{SE^3},$$

where λ is a constant. Again, since SEK is equally inclined to the spherical surface at the two points of intersection, we have (§§ 85, 86)

$$E=\frac{SE^2}{SK^2}\cdot K=\frac{SE^2}{SK^2}\cdot\frac{2a\omega . TK^2}{KK'};$$

and hence

$$F=\frac{\dfrac{\lambda}{SE^3}\cdot\dfrac{SE^2}{SK^2}\cdot\dfrac{2a\omega . TK^2}{KK'}}{EP^2}=\lambda . \frac{2a}{KK'}\cdot\frac{TK^2}{SE . SK^2 . EP^2}\cdot\omega.$$

Now, by considering the great circle in which the sphere is cut by a plane through the line SK, we find that

$$\left.\begin{array}{ll}\text{(fig. 1.)} & SK . SE=f^2-a^2\\ \text{(fig. 2.)} & KS . SE=a^2-f^2\end{array}\right\} \qquad (2),$$

and hence $SK . SE = SP . ST$, from which we infer that the triangles KST, PSE are similar; so that $TK : SK :: PE : SP$. Hence

$$\frac{TK^2}{SK^2 . PE^2}=\frac{1}{SP^2},$$

and the expression for F becomes

$$F=\lambda . \frac{2a}{KK'}\cdot\frac{1}{SE . SP^2}\cdot\omega \qquad (3).$$

Modifying this by (2) we have

$$\left.\begin{array}{ll}(\text{fig. 1.}) & F=\lambda.\dfrac{2a}{KK'}\cdot\dfrac{\omega}{(f^2-a^2)SP^2}\cdot SK \\ (\text{fig. 2.}) & F=\lambda\,\dfrac{2a}{KK'}\cdot\dfrac{\omega}{(a^2-f^2)SP^2}\cdot KS\end{array}\right\}\quad(4).$$

Similarly, if F' denote the attraction of E' on P, we have

$$(\text{fig. 1.})\quad F'=\lambda\frac{2a}{KK'}\cdot\frac{\omega}{(f^2-a^2)SP^2}\cdot SK',$$

$$(\text{fig. 2.})\quad F'=\lambda\frac{2a}{KK'}\cdot\frac{\omega}{(a^2-f^2)SP^2}\cdot K'S.$$

Now in the triangles which have been shown to be similar, the angles TKS, EPS are equal; and the same may be proved of the angles $TK'S$, $E'PS$. Hence the two sides SK, SK' of the triangle KSK' are inclined to the third at the same angles as those between the line PS and directions PE, PE' of the two forces on the point P; and the sides SK, SK' are to one another as the forces, F, F', in the directions PE, PE'. It follows, by "the triangle of forces," that the resultant of F and F' is along PS, and that it bears to the component forces the same ratios as the side KK' of the triangle bears to the other two sides. Hence the resultant force due to the two elements E and E', on the point P, is towards S, and is equal to

$$\lambda.\frac{2a}{KK'}\cdot\frac{\omega}{(f^2\sim a^2).SP^2}\cdot KK',\ \text{or}\ \frac{\lambda.2a.\omega}{(f^2\sim a^2)SP^2}.$$

The total resultant force will consequently be towards S; and we find, by summation (§ 83) for its magnitude,

$$\frac{\lambda.4\pi a}{(f^2\sim a^2)SP^2}.$$

Hence we infer that the resultant force at any point P, separated from S by the spherical surface, is the same as if a quantity of matter equal to $\dfrac{\lambda.4\pi a}{f^2\sim a^2}$ were concentrated at the point S.

91. To find the attraction when S and P are either both without or both within the spherical surface.

Take in CS (fig. 3.), or in CS produced through S (fig. 4.), a point S_1, such that

$$CS.CS_1=a^2.$$

Then, by a well-known geometrical theorem (see note on § 87), if E be any point on the spherical surface, we have

$$\frac{SE}{S_1E} = \frac{f}{a}.$$

Hence we have
$$\frac{\lambda}{SE^3} = \frac{\lambda a^3}{f^3 . S_1E^3}.$$

Hence, ρ being the electrical density at E, we have

$$\rho = \frac{\frac{\lambda a^3}{f^3}}{S_1E^3} = \frac{\lambda_1}{S_1E^3},$$

if
$$\lambda_1 = \frac{\lambda a^3}{f^3}.$$

Hence, by the investigation in the preceding paragraph, the attraction on P is towards S_1, and is the same as if a quantity of matter equal to $\dfrac{\lambda_1 . 4\pi a}{f_1^2 \sim a^2}$ were concentrated at that point; f_1 being taken to denote CS_1. If for f_1 and λ_1 we substitute their values, $\dfrac{a^2}{f}$ and $\dfrac{\lambda a^3}{f^3}$, we have the modified expression

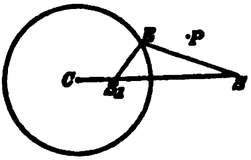


FIG. 3.

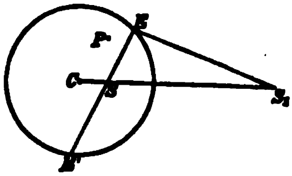


FIG. 4.

$$\frac{\lambda \frac{a}{f} . 4\pi a}{a^2 \sim f^2}$$

for the quantity of matter which we must conceive to be collected at S_1.

92. PROP. If a spherical surface be electrified in such a way that the electrical density varies inversely as the cube of the distance from an internal point S (fig. 4), or from the corresponding external point S_1, it will attract any external point, as if its whole mass were concentrated at S, and any internal

point as if a quantity of matter greater than the whole mass in the ratio of a to f were concentrated at S_1.
where f is [illegible] in fig. 4.

Let the density at E be denoted, as before, by $\frac{\lambda}{SE^3}$. Then, if we consider two opposite elements at E and E' which subtend a solid angle ω at the point S, the areas of these elements being (§ 96) $\frac{\omega.2a.SE^3}{E'E}$ and $\frac{\omega.2a.SE'^3}{E'E}$, the quantity of electricity which they possess will be

$$\frac{\lambda.2a.\omega}{E'E}\left(\frac{1}{SE}+\frac{1}{E'S}\right) \text{ or } \frac{\lambda.2a.\omega}{SE.E'S}.$$

Now $SE.E'S$ is constant (Euc. III. 35), and its value is a^2-f^2. Hence, by summation, we find for the total quantity of electricity on the spherical surface

$$\frac{\lambda.4\pi a}{a^2-f^2}.$$

Hence, if this be denoted by m, the expressions in the preceding paragraphs, for the quantities of electricity which we must suppose to be concentrated at the point S or S_1, according as P is without or within the spherical surface, become respectively

$$m, \text{ and } \frac{a}{f}m. \qquad \text{Q. E. D.}$$

Application of the preceding Theorems to the Problem of Electrical Influence.

93. PROB. To find the electrical density at any point of an insulated conducting sphere (radius a) charged with a quantity Q (either positive, or negative, or zero) of electricity, and placed with its centre at a given distance f from an electrical point M possessing m units of electricity.

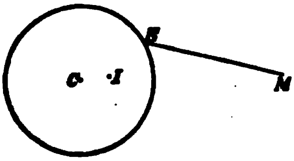
If the expression for the electrical density at any point E of the surface be

$$\rho=\frac{\lambda}{ME^3}+k \qquad (a),$$

λ and k being constants; the force exerted by the electrified surface on any internal point will be the same as if the constant distribution k, which (§ 78) exerts no force on an internal point, were removed; and therefore (§ 90) will be the same as if a quantity of matter equal to $\frac{\lambda.4\pi a}{f^2-a^2}$ were collected at the point M. Hence, if the condition

$$\frac{\lambda.4\pi a}{f^2-a^2}=-m \qquad (b)$$

be satisfied, the total attraction on an internal point, due to the electrified surface and to the influencing point, will vanish.



Hence this distribution satisfies the condition of equilibrium (§ 72); and to complete the solution of the proposed problem it only remains to determine the quantity k, so that the total quantity of electricity on the surface may have the given value Q. Now (§ 92) the total mass of the distribution, depending on the term $\frac{\lambda}{ME^3}$ in the expression for the density, since M is an *external* point, is equal to

$$\frac{a}{f}\cdot\frac{\lambda.4\pi a}{f^2-a^2}.$$

Hence, adding $4\pi a^2k$, the quantity depending on the constant term k, we obtain the entire quantity, which must be equal to Q; and we therefore have the equation

$$\frac{a}{f}\cdot\frac{\lambda.4\pi a}{f^2-a^2}+4\pi a^2k=Q \qquad (c).$$

From equations (b) and (c) we deduce

$$\lambda=-\frac{(f^2-a^2)m}{4\pi a} \text{ and } k=\frac{Q+\frac{a}{f}m}{4\pi a^2}.$$

Hence, by substituting in (a), we have

$$\rho=-\frac{(f^2-a^2)m}{4\pi a}\cdot\frac{1}{ME^3}+\frac{Q+\frac{a}{f}m}{4\pi a^2} \qquad (A),$$

as the expression of the required distribution of electricity. This agrees with the result obtained by Poisson, by means of an investigation in which the analysis known as that of "Laplace's coefficients," is employed.

94. To find the attraction exerted by the electrified conductor on any external point.

We may consider separately the distributions corresponding to the constant and the variable term in the expression for the electrical density at any point of the surface. The attraction of the first of these on an external point is (§ 87) the same as if its whole mass were collected at the centre of the sphere: the attraction of the second on an external point is (§ 92) the same as if its whole mass were collected at an interior point I, taken in MC so that $MI.MC=a^2$. Hence, according to the investigation in the preceding paragraph, we infer that the conductor attracts any external point with the same force as would be produced by quantities $Q+\frac{a}{f}m$, and $-\frac{a}{f}m$ of electricity, concentrated at the points C and I respectively.

COR. The resultant force at an external point infinitely near the surface is in the direction of the normal, and is equal to $4\pi\rho$, if ρ be the electrical density of the surface, in the neighbourhood.

95. To find the mutual attraction or repulsion between the influencing point, M, and the conducting sphere.

According to what precedes, the required attraction or repulsion will be the entire force exerted upon m units of electricity at the point M, by $Q+\frac{a}{f}m$ at C and $-\frac{a}{f}m$ at a point I, taken in CM, at a distance $\frac{a^2}{f}$ from C. Hence, if the required *attraction* be denoted by F (a quantity which will be negative if the actual force be of *repulsion*), we have

$$F=-\frac{\left(Q+\frac{a}{f}m\right)m}{f^2}+\frac{\frac{a}{f}m.m}{\left(f-\frac{a^2}{f}\right)^2} \qquad (B),$$

$$=\frac{a\{f^4-(f^2-a^2)^2\}m^2-f(f^2-a^2)^2Qm}{f^3(f^2-a^2)^2},$$

or

$$F=\frac{a^3(2f^2-a^2)m^2-f(f^2-a^2)^2Qm}{f^3(f^2-a^2)^2} \qquad (C).$$

COR. 1. If Q be zero or negative, the value of F is necessarily positive, since f must be greater than a; and therefore there is a force of attraction between the influencing point and the conducting sphere, whatever be the distance between them.

COR. 2. If Q be positive, then for sufficiently large values of f, F is negative, while for values nearly equal to a, F is positive. Hence if an electrical point be brought into the neighbourhood of a similarly charged insulated sphere, and if it be held at a great distance, the mutual action will be repulsive; if it then be gradually moved towards the sphere, the repulsion, which will at first increase, will, after attaining a maximum value, begin to diminish till the electrical point is moved up to a certain distance where there will be no force either of attraction or repulsion; if it be brought still nearer to the conductor, the action will become attractive and will continually augment as the distance is diminished.

If the value of Q be positive, and sufficiently great, a spark will be produced between the nearest part of the conductor and the influencing point, before the force becomes changed from repulsion to attraction.

ST. PETER'S COLLEGE,
July 7, 1848.

EFFECTS OF ELECTRICAL INFLUENCE ON INTERNAL SPHERICAL, AND ON PLANE CONDUCTING SURFACES.

96. In the preceding articles of this series certain problems with reference to conductors bounded externally by spherical surfaces have been considered. It is now proposed to exhibit the solutions of similar problems with reference to the distribution of electricity on concave spherical surfaces, and on planes.

The object of the following short digression is to define and explain the precise signification of certain technical terms and expressions which will be used in this and in subsequent papers on the Mathematical Theory of Electricity.

External and Internal Conducting Surfaces.

97. DEF. 1. A closed surface separating conducting matter

within it from air* without it, is called an *external conducting surface.*

DEF. 2. A closed surface separating air within it from conducting matter without it is called an *internal conducting surface.*

Thus, according to these definitions, a solid conductor has only one "conducting surface," and that "an external conducting surface."

A conductor containing within it one or more hollow spaces filled with air, possesses two or more "conducting surfaces;" namely, one "external conducting surface," and one or more "internal conducting surfaces."

A complex arrangement, consisting of a hollow conductor and other conductors insulated within it, presents several external and internal conducting surfaces; namely, an "external conducting surface" for each individual conductor, and as many "internal conducting surfaces" as there are hollow spaces in the different conductors.

98. In any arrangement such as this, there are different masses of air which are completely separated from one another by conducting matter. Now among the General Theorems alluded to in § 73, it will be proved that the bounding surface or surfaces of any such mass of air cannot experience any electrical influence from the surfaces of the other masses of air, or from any electrified bodies within them. Hence any statical phenomena of electricity which may be produced in a hollow space surrounded continuously by conducting matter,—whether this conducting envelope be a sheet even as thin as gold leaf, or a massive conductor of any external form and dimensions,—will depend solely on the form of the internal conducting surface.

99. PROP. *An internal conducting surface cannot receive a charge of electricity independently of the influence of electrified bodies within it.*

100. The demonstration of this proposition depends on what precedes, and on one of the General Theorems, already alluded to (§ 73), by which it appears that it is impossible to distribute a charge of electricity on a closed surface in such a manner that

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* See § 70, excluding all non-conductors except air, or gases.

there may be no resultant force exerted on external points, and consequently impossible, with merely a distribution of electricity on an internal conducting surface, to satisfy the condition of electrical equilibrium with reference to the conducting matter which surrounds it.

The preceding proposition (§ 99) is fully confirmed by experiment (Faraday's *Experimental Researches*, §§ 1173, 1174). In fact, the certainty with which its truth has been practically demonstrated in a vast variety of cases, by all electrical experimenters, may be regarded as a very strong part of the evidence on which the Elementary Laws as stated above (§ 72) rest.

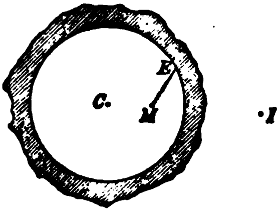
101. It might be further stated that the total quantity of electricity produced by influence on an internal conducting surface is necessarily equal in every case to the total quantity of electricity on the influencing electrified bodies insulated within it. This will also be demonstrated among the General Theorems; but its truth in the special case which we are now to consider, will, as we shall see, be established by a special demonstration.

Electrical Influence on an Internal Spherical Conducting Surface.

102. In investigating the effects of electrical influence upon an external, or convex, spherical conducting surface (§§ 93, 94, 95), we have considered the conductor to be insulated and initially charged with a given amount of electricity. In the present investigation no such considerations are necessary, since, according to the statements in the preceding paragraphs, it is of no consequence, in the case now contemplated, whether the conductor containing the internal conducting surface be insulated or not; and it is impossible to charge this internal surface initially, or to charge it at all, independently of the influence of electrified bodies within it. With the modifications and omissions necessary on this account, the preceding investigations are applicable to the case now to be considered.

103. PROB. To find the electrical density at any point of an internal spherical conducting surface with an electrical point insulated within it.

Let m denote the quantity of electricity in the electrical point M; f its distance from C the centre of the sphere, and a the radius of the sphere.



If the expression for the electrical density at any point E of the internal surface be

$$\rho = \frac{\lambda}{ME^3},^*$$

(λ a constant); the force exerted by the electrified spherical surface on any point without it will (§ 90) be the same as if a quantity of matter equal to $\frac{\lambda . 4\pi a}{a^2 - f^2}$ were collected at the point M. Hence if we take λ such that

$$\frac{\lambda . 4\pi a}{a^2 - f^2} = -m,$$

the total resultant force, due to the given electrical point and to the electrified surface, will vanish at every point external to the spherical surface, and consequently at every point within the substance of the conductor; so that the condition of electrical equilibrium (§ 72), in the prescribed circumstances, is satisfied. We conclude, therefore, that the required density at any point E, of the internal spherical surface is given by the equation

$$\rho = -\frac{(a^2 - f^2)m}{4\pi a} \cdot \frac{1}{ME^3} \qquad (A).$$

This solution of the problem is complete, since it satisfies all the conditions that can possibly be prescribed, and it is unique, as follows from the general Theorem referred to in § 73.†

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[*] We cannot here, as in (*a*) of § 93, annex a constant term, since in this case there would result a force due to a corresponding quantity of electricity, concentrated at the centre of the sphere on all points of the conducting mass.

[†] For if there were two distinct solutions there would be two different distributions on the spherical surface, each balancing on external points the action of the internal influencing body, and therefore each producing the same force at external points. Hence a distribution, in which the electrical density at each point is equal to the difference of the electrical densities in those two, would produce no force at external points. But, by the theorem alluded to, no distribution on a closed surface of any form can have the property of producing no force on external points; and therefore the hypothesis that there are two distinct solutions is impossible.

The theorem made use of in this reasoning is susceptible of special *analytical*

COR. The total quantity of electricity produced by the influence of an electrical point within an internal spherical conducting surface is equal, but of the opposite kind to that of the influencing point.

This follows at once from the investigation of § 92; from which we also deduce the conclusion stated below in the next section.

104. The entire electrical force, which vanishes for all points external to the conducting surface, may, for points within it, be found by compounding the force due to the given influencing point M (charged, by hypothesis, with a quantity m of electricity) with that due to an imaginary point I, taken in CM produced, at such a distance from C that $CM.CI = a^2$, and charged with a quantity of electricity equal to $-\frac{a}{f}m$.

COR. The resultant force at an internal point infinitely near the surface, is in the direction of the normal, and is equal to $4\pi\rho$, if ρ be the electrical density of the surface in the neighbourhood.

105. The mutual attraction between the influencing point M, and the surface inductively electrified will be found as in § 95, provided the *uniform supplementary distribution* which was there introduced be omitted. Hence, omitting the term of (B) which depends on this supplementary distribution; or simply, without reference to (B), considering the mutual force between m at M and $-\frac{a}{f}m$ at I, a force which is necessarily attractive as the two electrical points M and I possess opposite kinds of electricity; we obtain

$$F = \frac{\frac{a}{f}m.m}{\left(\frac{a^2}{f} - f\right)^2} = \frac{afm^2}{(a^2 - f^2)^2} \qquad (B)$$

as the expression for the required attraction.

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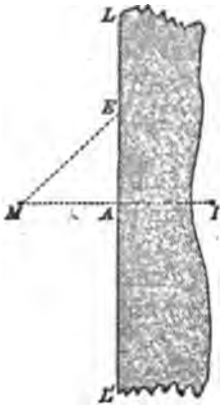
demonstration (with the aid of the method in which "Laplace's coefficients" are employed) for the case of a spherical surface; but such an investigation would be inconsistent with the synthetical character of the present series of papers, and I therefore do no more at present than allude to the general theorem.

Electrical Influence on a Plane Conducting Surface of infinite extent.

106. If, in either the case of an external or the case of an internal spherical conducting surface, the radius of the sphere be taken infinitely great, the results will be applicable to the present case of an infinite plane; and it is clear that from either we may deduce the complete solution of the problem of determining the distribution of electricity, produced upon a conducting plane, by the influence of an electrical point. The "supplementary distribution," which, in the case of a convex spherical conducting surface, must in general be taken into account, will, in the case of a sphere of infinite radius, be a finite quantity of electricity uniformly distributed over a surface of infinite extent, and will therefore produce no effect; and the same results will be obtained whether we deduce them from the case of an external or of an internal spherical surface.

107. Let M be an electrical point possessing a quantity m of electricity placed in the neighbourhood of a conductor bounded on the side next M by a plane LL' which we must conceive to be indefinitely extended in every direction; it is required to determine the electrical density at any point E of the conducting surface.



Draw MA perpendicular to the plane, and let its length be denoted by p. We may, in the first place, conceive that instead of the plane surface we have a spherical conducting surface entirely enclosing the air in which M is insulated; and, supposing the shortest line from M to the spherical surface to be equal to p, we should have, according to the notation of § 103,

$$f = a - p.$$

Hence the expression (A) becomes

$$\rho = -\frac{2ap-p^2}{4\pi a} \cdot \frac{m}{ME^3} = -\left(\frac{p}{2\pi} - \frac{p^2}{4\pi a}\right)\frac{m}{ME^3}.$$

In this, let a be supposed to be infinitely great; the second term within the vinculum will vanish, and we shall have simply

$$\rho = -\frac{mp}{2\pi ME^3} \qquad (A)$$

for the required electrical density at the point E of the infinite plane electrified inductively through the influence of the point M.

COR. The total amount of the electricity produced by induction is equal in quantity, but opposite in kind, to that of the influencing point M. We have seen already that the same proposition is true in general for internal spherical surfaces inductively electrified; but it does not hold for an external spherical surface, even if we neglect the "supplementary distribution," as it appears from the demonstration of § 92, that the amount of the distribution expressed by the first term (that which varies inversely as the cube of the distance from the influencing point) of the value of ρ in equation (A) of § 93, is equal to $-\frac{a}{f}m$. The infinite plane may, as we have seen, be regarded as an extreme case of either an external or an internal spherical surface; and the proposition which is in general true for internal, but not true for external spherical surfaces, holds in this limiting intermediate case.

108. To determine the resultant force at any point in the air, before the conducting plane, it will be only necessary, as in § 104, to compound the action of the given electrical point with that of an imaginary point I.

To find this point, we must produce MA beyond A to a distance AI, determined by the equation $CM.CI = a^2$; which, if we denote AI by p', becomes $(a-p)(a+p') = a^2$.

From this we deduce

$$p' = \frac{ap}{a-p} = \frac{p}{1-\frac{p}{a}},$$

and thence, in the case of $a = \infty$, we deduce

$$p' = p.$$

Again, for the quantity of electricity to be concentrated at I, we have the expression

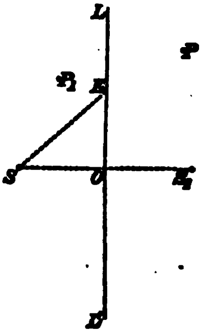
$$m' = -\frac{a}{a-p}m, \text{ or, when } a = \infty,\ m' = -m.$$

Hence the force at any point before the plane will be obtained by compounding that due to the given electrical point M, with a force due to an imaginary point I, possessing an equal quantity of the other hand of electricity, and placed at an equal distance behind the plane in the perpendicular MA produced.

109. If reference be made to the general demonstration (§ 90) on which all the special conclusions with reference to the effects of electrical influence on convex, concave, or plane conducting surfaces depend, we see that the geometrical construction employed fails in the case of a sphere of infinite radius, becoming nugatory in almost every step: we have however deduced *conclusions* which are not nugatory, but, on the contrary, assume a remarkably simple form for this case; and we may regard as rigorously established the solution of the problem of electrical influence on an infinite plane which has been thus obtained.

110. It is interesting to examine the nugatory forms which occur in attempting to apply the demonstrations of §§ 90 and 92, to the case of an infinite plane; and it is not difficult to derive a special demonstration, free from all nugatory steps, of the following proposition.

Let LL' be an infinite "material plane," of which the "density" in different positions varies inversely as the cube of the distance from a point S, or from an equidistant point S_1, on the other side of the plane. The resultant force at any point P is the same as if the whole matter of the plane were concentrated at S; and the resultant force at any point P_1, on the other side of the plane, is the same as if the whole matter were collected at S_1.



111. In the course of the demonstration (in that part which corresponds to the investigation in § 93) it would appear that, if the density at any point E of the plane is given by the expression

$$\rho = \frac{\lambda}{SE^3},$$

the entire quantity of matter distributed over the infinite extent of the plane is given by the expression

$$-m=\frac{2\pi\lambda}{p}.$$

This proposition and that which precedes it* contain the simplest expression of the mathematical truths on which the solution of the problem of electrical influence on an infinite plane depends, and we might at once obtain from them the results given above. For an isolated investigation of this case of electrical equilibrium, this would be a better form of solution: but I have preferred the method given above, since the solution of the more general problem, of which it is a particular case, had been previously given.

112. The case of electrical influence which has been considered

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* The two propositions may be analytically expressed as follows:—

Let O, the point in which SS_1 cuts the plane, be origin of co-ordinates, and let this line be axis of z. Then, taking OX, OY in the plane, let the co-ordinates of P be (x, y, z). Let also those of E be $(\xi, \eta, 0)$; so that we have

$$\rho=\frac{\lambda}{(\xi^2+\eta^2+p^2)^{\frac{3}{2}}}.$$

Hence the proposition stated in the text (§ 111), that the entire quantity of matter distributed over the infinite extent of the plane is equal to $\frac{2\pi\lambda}{p}$, is thus expressed:—

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\lambda d\xi d\eta}{(\xi^2+\eta^2+p^2)^{\frac{3}{2}}}=\frac{2\pi\lambda}{p}.$$

This equation may be very easily verified, and so an extremely simple analytical demonstration of one of the theorems enunciated above is obtained.

Again, the proposition with reference to the attraction of the plane may, according to the well-known method, be expressed most simply by means of the potential. This must, in virtue of the enunciation in § 110, be equal to the potential due to the same quantity of matter, collected at the point S, or the point S_1, according as the attracted point is separated from the former or from the latter by the plane. Hence we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\lambda d\xi d\eta}{(\xi^2+\eta^2+p^2)^{\frac{3}{2}}\{(x-\xi)^2+(y-\eta)^2+z^2\}^{\frac{1}{2}}}=\frac{\frac{2\pi\lambda}{p}}{\{x^2+y^2+(\pm z+p)^2\}^{\frac{1}{2}}},$$

the positive or negative sign being attached to z in the denominator of the second member, according as z is given with a positive or negative value. This equation (of which a geometrical demonstration is included in §§ 107 and 108, in connexion with § 90) is included in a result (the evaluation of a certain multiple integral), of which three different analytical demonstrations were given in a paper *On certain Definite Integrals suggested by Problems in the Theory of Electricity*, published in March 1847 in this *Journal*, vol. ii. p. 109 (IX. below).

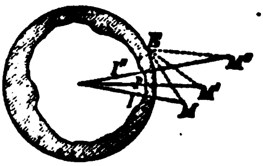
might at first sight appear to be of a singularly unpractical nature, since a conductor presenting on one side a plane surface of infinite extent in every direction would be required for fully realizing the prescribed circumstances. If, however, we have a plane table of conducting matter, or covered with a sheet of tinfoil, or if we have a wall presenting an uninterrupted plane surface of some extent, the imagined circumstances are, as we readily see, *approximately* realized with reference to the influence of any electrical point in the neighbourhood of such a conducting plane, provided the distance of the influencing point from the plane be small compared with its distance from the nearest part where the continuity of the plane surface is in any way broken.

Fortbreda, Belfast, *Oct.* 17, 1849.

INSULATED SPHERE SUBJECT TO THE INFLUENCE OF A BODY OF ANY FORM ELECTRIFIED IN ANY GIVEN MANNER.

113. The problem of determining the distribution of electricity upon a sphere, or upon internal or plane spherical conducting surfaces, under the influence of an electrical point, was fully solved in §§ 89...112 of this series of papers. On the principle of the superposition of electrical forces (§ 63) we may apply the same method to the solution of corresponding problems with reference to the influence of any number of given electrical points.

114. Thus let M, M', M'' be any number of electrical points possessing respectively m, m', m'' units of electricity, at distances f, f', f'' from C the centre of a sphere insulated and charged with a quantity Q of electricity. The actual distribution of electricity on the spherical surface must be such that the force due to it at any internal point shall be equal and opposite to the force due to the electricity at M, M', M''. Now if there were a distribution of electricity on the spherical surface such that the density at any point E would be $\frac{\lambda}{ME^2}$, the force due to this at any internal point



would (§ 90) be the same as that due to a quantity $\frac{\lambda.4\pi a}{f^2-a^2}$ concentrated at the point M; and therefore if we take

$$\lambda=-\frac{(f^2-a^2)m}{4\pi a},$$

the force at internal points due to this distribution would be equal and opposite to the force due to the actual electricity of M. We might similarly express distributions which would respectively balance the actions of M', M'', etc., upon points within the sphere; and thence, by supposing all those distributions to coexist on the surface, we infer that a single distribution such that the density at E is equal to

$$-\left\{\frac{(f^2-a^2)m}{4\pi a}\frac{1}{ME^3}+\frac{(f'^2-a^2)m'}{4\pi a}\frac{1}{M'E^3}+\frac{(f''^2-a^2)m''}{4\pi a}\frac{1}{M''E^3}\right\}$$

would balance the joint action of all the electrical points M, M', M'', on points within the sphere. Again, from § 92, we infer that the total quantity of electricity in such a distribution is

$$-\left(\frac{a}{f}m+\frac{a}{f'}m'+\frac{a}{f''}m''\right).$$

Hence, unless the data chance to be such that Q is equal to this quantity, a supplementary distribution will be necessary to constitute the actual distribution which it is required to find. The amount of this supplementary distribution will be

$$Q+\frac{a}{f}m+\frac{a}{f'}m'+\frac{a}{f''}m'';$$

which must be so distributed as to produce no force on internal points.

115. Taking then the distribution found above, which balances the action of the electricity at M, M', etc., on points within the sphere, and a uniform supplementary distribution; and superimposing one on the other, we obtain a resultant electrical distribution in which the density at any point E of the surface of the sphere is given by the equation

$$\rho=-\left\{\frac{(f^2-a^2)m}{4\pi a}\frac{1}{ME^3}+\frac{(f'^2-a^2)}{4\pi a}\frac{1}{M'E^3}+\text{etc.}\right\}+\frac{Q+\frac{a}{f}m+\frac{a}{f'}m'+\text{etc.}}{4\pi a^2} \qquad (1);$$

and we draw the following conclusions:—

(1.) The total force at any internal point, due to this distribution and to the electricity of M, M', etc., vanishes.

(2.) The entire quantity of electricity on the spherical surface is equal to Q.

Hence this distribution of the given charge on the sphere satisfies the condition of electrical equilibrium under the influence of the given electrical points M, M', etc.; and (§ 73) it is therefore the distribution which actually exists upon the spherical conductor in the prescribed circumstances.

116. The resultant force at any external point may be found as in the particular case treated in § 94. Thus, if we join MC, $M'C$, $M''C$, and take in the lines so drawn, points I, I', I'' respectively, at distances from C such that

$$CI.CM = CI'.CM' = CI''.CM'' = a^2,$$

the resultant action due to the actual electricity of the spherical surface will, at any external point, be the same as if the sphere were removed, and electrical points I, I', etc., substituted in its stead, besides (except in the case when the supplementary distribution vanishes) an electrical point at C: and the quantities of electricity which must be conceived for this representation, to be concentrated at these points, are respectively

$$\left.\begin{array}{ll} -\frac{a}{f}m, & \text{at } I; \\ -\frac{a}{f}m', & \text{at } I'; \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \text{and} \quad Q+\frac{am}{f}+\frac{am'}{f}+\text{etc., at } C. & \end{array}\right\} \quad (2).$$

117. By means of these imaginary electrical points we may give another form to the expression for the distribution on the spherical surface, which in many important cases, especially that of two mutually influencing spherical surfaces, is extremely convenient. For (as in § 94, Cor.) it is readily seen that the first term, in the expression for ρ multiplied by 4π, or

$$-\frac{(f^2-a^2)m}{a}\frac{1}{ME^3},$$

is the resultant force at E, due to M and I, and that this force is in the direction of a normal to the spherical surface through

E; and that similar conclusions hold with reference to the other similar terms of (2). Again, the last term,

$$\frac{Q+\frac{a}{f}m+\frac{a}{f'}m'+\text{etc.}}{4\pi a^2}$$

is the expression for the force at E, due to the imaginary electric point C, divided by 4π; and this force also is in the direction of the normal. Hence, with reference to the total resultant action at E, due to M, M', etc., and the spherical surface, or the imaginary electrical points within it, we infer

(1.) That this force is in the direction of the normal;

(2.) That if R be its magnitude considered as positive or negative according as it is from or towards the centre of the sphere, and ρ the electrical density at E, we have

$$\rho=\frac{1}{4\pi}R \qquad (3).$$

These two propositions constitute the expression, for the case of a spherical conductor subject to any electric influence, of *Coulomb's Theorem.*[*]

118. The total action exerted by the given electrical points, and by the sphere with its electricity disturbed by their influence upon a given electrified body placed anywhere in their neighbourhood, might, as we have seen, be found by substituting in place of the sphere the group of electrical points which represents its external action, provided there were no disturbance produced by the influence of this electrified body. This hypothesis, however, cannot be true unless the sphere, after experiencing as a conductor the influence of M, M', etc., were to become a non-conductor so as to preserve with *rigidity* the distribution of its electricity when the new electrified body is brought into its neighbourhood: and consequently, when it is asserted that the resultant force *at* any external point P is due to the group of electrical points determined in the preceding paragraphs, we must remember that the disturbing influence that would be actually exerted upon the distribution on the

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[*] For a general demonstration of this theorem, virtually the same as the original demonstration given by Coulomb himself, see *Cambridge Mathematical Journal* (1842) vol. iii. p. 75 (or §§ 7, 8, above).

spherical surface by a unit of electricity at the point P, is excluded in the definition (§ 65) of the expression "*the resultant electrical force at a point.*"

119. The actual force exerted upon any one, M, of the influencing points may be determined by investigating the resultant force at M, due to all the others and to the conductor, and multiplying it by the quantity of electricity, m, situated at this point, since in this case the influence of the body on which the force is required has been actually taken into account.

120. It follows that the entire mutual action between all the given electrical points and the sphere under their influence is the same as the mutual action between the two systems of electrical points,

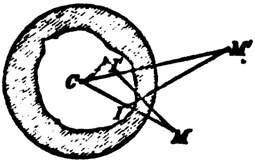
$$\left.\begin{array}{l} m \text{ at } M \\ m' \text{ at } M' \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \\ \end{array}\right\} \text{ and } \left\{\begin{array}{ll} -\frac{a}{f}m & \text{at } I \\ -\frac{a}{f}m' & \text{at } I' \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \multicolumn{2}{l}{Q+\frac{a}{f}m+\frac{a}{f}m'+\text{etc., at } C.} \end{array}\right.$$

This action may be fully determined with any assigned *data*, by the elementary principles of statics.

121. There is a remarkable characteristic of this resultant action which ought not to be passed over, as it is related to a very important physical principle of symmetry, of which many other illustrations occur in the theories of electricity and magnetism. It is expressed in the following proposition:—

The mutual action between a spherical conductor and any given electrified body consists of a single force in a line through the centre of the sphere.

Let us conceive the given electrified body either to consist of a group of electrical points, or to be divided into infinitely small parts, each of which may be regarded as an electrical point. The mutual action between the given body and the conducting sphere under its influence is therefore to be found by compounding all the forces between the points M, M', etc., of the given body, and the points I, I', etc. . . . . and C, of the imaginary system within the sphere determined by the construction and formulæ of the preceding paragraphs. Of these, the forces between M and C, between M' and C', etc.; and again, between M and I, between M' and I', etc., are actually in lines passing through C; and, therefore, if there were no other forces to be taken into account the proposition would be proved. But we have also a set of forces between M and I', between M and I'', etc., none of which, except in particular cases, are in lines through C, and, therefore, it remains for us to determine the nature of the resultant action of all these forces. For this purpose let us consider any two points M, M' of the given influencing body and the corresponding imaginary points I, I'; and let us take the force between M and I', and along with it the force between I and M'.



These two forces lie in the plane MCM', since, by the construction given above, I and I' are respectively in the lines CM and CM'; and hence they have a single resultant. Now the force in MI' is due to m units of electricity at M, and $-\frac{a}{f'}m'$ units at I'; and (§ 64) it is therefore a force of repulsion equal to

$$\frac{-\frac{a}{f'}m'.m}{I'M^2}, \text{ or a force of attraction equal to } \frac{\frac{a}{f'}m'.m}{I'M^2}.$$

Similarly, we find
$$\frac{\frac{a}{f}m.m'}{IM'^2}$$
for the attraction between M' and I. Now since, by construction, $CM.CI = CM'.CI'$, the triangles $I'MC$, $IM'C$, which have a common angle at C, are similar. Hence

$$\frac{I'M^2}{IM'^2} = \frac{CI'.CM}{CI.CM'} = \frac{\frac{a^2}{f'}.f}{\frac{a^2}{f}.f'},$$

from which we deduce

$$\frac{\frac{a}{f'}m'.m}{I'M^2}f = \frac{\frac{a}{f}m.m'}{IM'^2}f'.$$

Now if we multiply the first member of this equation by $\sin CMI'$, we obtain the moment round C of the force between

I' and M; and similarly, by multiplying the second member by sin $CM'I$, we find the moment of the force between M' and I; and, since the angle at M is equal to the angle at M', we infer that the moments of the two forces round C are equal. From this it follows that the resultant of the forces in MI' and $M'I$ is a force in a line passing through C. Now the entire group of forces between points of the given body and *non-correspondent* imaginary points, consists of pairs such as that which we have just been considering; and therefore the mutual action is the resultant of a number of forces in lines passing through C. This, compounded with the forces between M, M', etc., and the *corresponding* imaginary points, and the forces between M, M', etc., and the imaginary electrical point at C, gives for the total mutual action a final resultant in a line passing through C.

122. It follows from this theorem that if a spherical conductor be supported in such a manner as to be able to turn freely round its centre, or round any axis passing through its centre, it will remain in equilibrium when subjected to the influence of any external electrified body or bodies. We may arrive at the same conclusion by merely considering the perfect symmetry of the sphere, round its centre or round any line through its centre, without assuming any specific results with reference to the distribution of electricity on spherical conductors. For if there were a tendency to turn round any diameter through the influence of external electrified bodies, the sphere would, on account of its symmetry, experience the same tendency when turned into any other position, its centre and the influencing bodies remaining fixed; and there would therefore result a continually accelerated motion of rotation. This being a physical impossibility, we conclude that the sphere can have no tendency to move when its centre is fixed, whatever be the electrical influence to which it is subjected.

123. It is very interesting to trace the different actions which, according to the synthetical solution of the problem of electrical influence investigated above, must balance to produce this equilibrium round the centre of a spherical conductor subjected to the influence of a group of electrical points. Let us, for example, consider the case of two influencing points.

For fixing the ideas, let us conceive the sphere to be capable of turning round a vertical axis, and let the influencing points be situated in the horizontal plane of its centre, C. If at first there be only one electrical point, M, which we may suppose to be positive, the sphere under its influence will be electrified with a distribution symmetrical round the line MC, but with more negative, or, as the case may be, less positive, electricity, on the hemisphere of the surface next M than on the remote hemisphere. If another positive electrical point, M', be brought into the neighbourhood of the sphere, on a level with its centre, and on one side or the other of MC, and if for a moment we conceive the sphere to be a perfect non-conductor of electricity; this second point, acting on the electricity as distributed under the influence of the first, will make the sphere tend to turn round its vertical axis. Thus if AA_1 be a diameter of the sphere in the line $MACA_1$, the sphere would tend to turn from its primitive position so as to bring the point A of its surface nearer M'. If now the sphere be supposed to become a perfect conductor, the distribution of its electricity will be altered so as to be no longer symmetrical round AA_1. This alteration we may conceive to consist of the superposition of a distribution of equal quantities of positive and negative electricities symmetrically distributed round the line $M'C$, with the negative electricity preponderating on the hemisphere nearest to M'. To obtain the total action of the two points on the electrified sphere, it will now be necessary to compound the action of M', and the action of M, on this superimposed distribution with the action previously considered. Of these the former consists of a simple force of attraction in the line $M'C$; but the latter, if referred to C the centre of the sphere, will give, besides a simple force, a couple round a vertical axis, tending to turn the sphere in such a direction as to bring the point A' of its surface nearer M. Now, as we know *a priori* that there can be no resultant tendency to turn arising from the entire action upon the sphere, it follows that the moment of this couple must be equal to the moment of the contrary couple, which, as we have seen previously, results from the action of M' on the sphere as primitively electrified under the influence of M. This is precisely the proposition of which a synthetical demonstration was given in § 121, and we accordingly see that that demonstration is merely the verification of a proposition of which the truth is rendered certain by *a priori* reasoning founded on general physical principles.

124. When the influencing body, instead of being, as we have hitherto conceived it, a finite group of isolated electrical points, is a continuous mass continuously electrified, we must imagine it to be divided into an infinite number of electrical points; and then, by means of the integral calculus, the expressions investigated above may be modified so as to be applicable to any conceivable case.

125. It appears from the considerations adduced in §§ 99, 100, that it is impossible to have an internal spherical conducting surface, or an infinite plane conducting surface, insulated and charged with a given amount of electricity; and that consequently, there being no "uniform supplementary distributions" to be taken into account, the solutions of ordinary problems with reference to such surfaces are somewhat simpler than those in which it may be proposed to consider an insulated conducting sphere possessing initially a given electrical charge. All the investigations of the present article, except those which have reference to the "supplementary distribution" and which are not required, are at once applicable to cases of internal or of plane conducting surfaces.

126. The importance of considering the imaginary electrical points I, I', etc. (and C, the centre of the sphere in the case of an external spherical surface), whether for solving problems with reference to the mutual forces called into action by the electrical excitation, or for determining the distribution of electricity on the spherical surface, has been shown in what precedes. Hence it will be useful, before going further in the subject, to examine the nature of such groups of imaginary points, when the influencing bodies are either finite groups of electrical points, or continuously electrified bodies. [See XIV. below, or Thomson and Tait's *Natural Philosophy*, §§ 512...518.]

127. The term *Electrical Images*, which will be applied to the imaginary electrical points or groups of electrical points, is suggested by the received language of Optics; and the close analogy of optical images will, it is hoped, be considered as a sufficient justification for the introduction of a new and extremely convenient mode of expression into the Theory of Electricity.

STOCKHOLM, *September* 20, 1849.

VI. ON THE MUTUAL ATTRACTION OR REPULSION BETWEEN TWO ELECTRIFIED SPHERICAL CONDUCTORS.

[*Philosophical Magazine*, April and August 1853.]

128. In a communication made to the British Association at Cambridge in 1845, I indicated a solution adapted for numerical calculation, of the problem of determining the mutual attraction between two electrified spherical conductors. A paper (II. above) published in November of the same year in the first Number of the *Cambridge and Dublin Mathematical Journal* contains a formula actually expressing the complete solution for the case of an insulated sphere and a non-insulated sphere of equal radius (§ 30, above), and numerical results calculated for four different distances for the sake of comparison with experimental results which had been published by Mr. Snow Harris. The investigation by which I had arrived at this solution, which was equally applicable to the general problem of finding the attraction between any two electrified spherical conductors, has not hitherto been published; but it was communicated in July 1849 to M. Liouville, along with another very different method by which I had just suceeded in arriving at the same result, in a letter the substance of which constitutes the present communication. Formulæ marked (8) . . . . (18) in that letter expressed the details of the solution according to the two methods. They are reproduced here in terms of the same notation, and with the same numbers affixed. The first-mentioned method is expressed by the formulæ (16), (17), (18), and the other by (8) . . . . (15). The formulæ marked with letters (*a*), (*b*), etc., in the present paper, express details of which I had not preserved exact memoranda.

129. Let A and B designate the two spherical conductors; let a and b be their radii, respectively; and let c be the distance between their centres. Let them be charged with such quantities of electricity, that, when no other conductors and no

excited electrics are near them, the values of the potential[*] within them may be u and v respectively.

130. The distribution of electricity on each surface may be determined with great facility by applying the "principle of successive influences" suggested by Murphy (Murphy's *Electricity*, Cambridge, 1833, p. 93), and determining the effect of each influence by the method of "electrical images," given in a paper entitled "Geometrical Investigations regarding Spherical Conductors."[†] The following statement shows as much as is required of the results of this investigation for our present purpose.

131. Let us imagine an electrical point containing a quantity of electricity equal to ua to be placed at the centre of A, and another vb at the centre of B. The image of the former in B will be $-\frac{b}{c}\cdot ua$, at a point in the line joining the centres, and distant by $\frac{b^2}{c}$ from the centre of B. The image of this in A will be $\frac{a}{c-\frac{b^2}{c}}\cdot\frac{b}{c}\cdot ua$, in the same line, at a distance $\frac{a^2}{c-\frac{b^2}{c}}$ from the centre of A; the image of this point in B will be $\frac{-b}{c-\frac{a^2}{c-\frac{b^2}{c}}}\cdot\frac{a}{c-\frac{b^2}{c}}\cdot\frac{b}{c}\cdot ua$, at a distance $\frac{b^2}{c-\frac{a^2}{c-\frac{b^2}{c}}}$ from the centre of B; and so on: and in a similar manner we may derive a series of imaginary points from vb at the centre of B. To specify completely these two series of imaginary points, let p_1, p'_1, p_2, p'_2, p_3, p'_3, etc., denote the masses of the series of which the first is at the centre of A; and let f_1, f'_1, f_2, f'_2, etc., denote the distances of these points from the centres of A and B alter-

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[*] The potential at any point in the neighbourhood of, or within, an electrified body, is the quantity of work that would be required to bring a unit of positive electricity from an infinite distance to that point, if the given distribution of electricity were maintained unaltered. Since the electrical force vanishes at every point within a conductor, the potential is constant throughout its interior.

[†] *Cambridge and Dublin Mathematical Journal*, Feb. 1850 (v. above, § 127).

nately; and, again, let $q_1, q'_1, q_2, q'_2, \ldots$, denote the masses, and $g_1, g'_1, g_2, g'_2, \ldots$, the distances of the successive points of the other series from the centres of B and A alternately. These quantities are determined by using the following equations, and giving n successively the values 1, 2, 3, ... :—

$$\left.\begin{cases} f_1=0, & p_1=ua \\ f'_n=\dfrac{b^2}{c-f_n}, & p'_n=-p_n\dfrac{f'_n}{b} \\ f_{n+1}=\dfrac{a^2}{c-f'_n}, & p_{n+1}=-p'_n\dfrac{f_{n+1}}{a} \end{cases}\right\} \left.\begin{cases} g_1=0, & q_1=vb \\ g'_n=\dfrac{a^2}{c-g_n}, & q'_n=-q_n\dfrac{g'_n}{a} \\ g_{n+1}=\dfrac{b^2}{c-g'_n}, & q_{n+1}=-q'_n\dfrac{g_{n+1}}{b} \end{cases}\right\} (8).$$

The two series of imaginary electrical points thus specified, would, if they existed, produce the same action in all space external to the spherical surfaces as the actual distributions of electricity do, those (p_1, q'_1, p_2, q'_2 etc.) which lie within the surface A producing the effect of the distribution on A, and the others (q_1, p'_1, q_2, p'_2, etc.), all within the surface B, the effect of the actual distribution on B. Hence the resultant force between the two partial groups is the same as the resultant force due to the mutual action between the actual distributions of electricity on the two conductors; and if this force, considered as positive or negative according as repulsion or attraction preponderates, be denoted by F, we have

$$F=\sum_{s=1}^{s=\infty}\sum_{t=1}^{t=\infty}\left\{\frac{p_s q_t}{(c-f_s-g_t)^2}+\frac{p'_s q'_t}{(c-f'_s-g'_t)^2}+\frac{p_s p'_t}{(c-f_s-f'_t)^2}+\frac{q_s q'_t}{(c-g_s-g'_t)^2}\right\} (9),$$

where $\Sigma\Sigma$ denotes a double summation, with reference to all integral values of s and t. The following process reduces this double series to the form of a single infinite series, of which the successive terms may be successively calculated numerically in any particular case with great ease.

132. First, taking from (8) expressions for p_s and f_s in terms of inferior order, and for q_t and g_t in terms of higher order, and continuing the reduction successively, we have

$$\frac{p_s q_t}{c-f_s-g_t}=\frac{p'_{s-1}\dfrac{a}{c-f'_{s-1}}\cdot q'_t\dfrac{a}{g'_t}}{c-\dfrac{a^2}{c-f'_{s-1}}-\left(c-\dfrac{a^2}{g'_t}\right)}=\frac{p'_{s-1}q'_t}{c-f'_{s-1}-g'_t}$$

$$=\frac{p_{s-1}\dfrac{b}{c-f_{s-1}}\cdot q_{t+1}\dfrac{b}{g_{t+1}}}{c-\dfrac{b^2}{c-f_{s-1}}-\left(c-\dfrac{b^2}{g_{t+1}}\right)}=\frac{p_{s-1}q_{t+1}}{c-f_{s-1}-g_{t+1}};$$

and therefore

$$\frac{p_s q_t}{c-f_s-g_t}=\frac{p_{s-1}q_{t+1}}{c-f_{s-1}-g_{t+1}}=\frac{p_{s-2}q_{t+2}}{c-f_{s-2}-g_{t+2}}\cdots=\frac{p_1 q_{t+s-1}}{c-f_1-g_{t+s-1}}=-uq'_{t+s-1}$$

and

$$\frac{p'_s q'_t}{c-f'_s-g'_t}=-uq'_{t+s}.$$

Similarly, we find

$$\frac{p_s p'_t}{c-f_s-f'_t}=\frac{p_{s-1}p'_{t+1}}{c-f_{s-1}-f'_{t+1}}=\ldots=\frac{p_1 p'_{t+s-1}}{c-f'_{t+s-1}}=-up_{t+s}$$

and

$$-\frac{q_s q'_t}{c-g_s-g'_t}=-vq_{t+s}.$$

Now $\frac{p'_n}{u}=\frac{q'_n}{v}$; and $\frac{p_n}{u}$ and $\frac{q_n}{v}$ are each independent of u and v; hence the following notation may be adopted conveniently:

$$\left.\begin{aligned}p'_n&=-\frac{u}{S_n},\quad q'_n=-\frac{v}{S_n}\\ p_n&=\frac{u}{P_n},\quad q_n=\frac{v}{Q_n}\end{aligned}\right\}\qquad(13).$$

Then, taking n to denote $t+s$ in the preceding equations, we have

$$\left.\begin{aligned}&\frac{p_{n-t}q_t}{c-f_{n-t}-g_t}=\frac{p'_{n-t-1}q'_t}{c-f'_{n-t-1}-g'_t}=\frac{uv}{S_{n-1}};\\ &\frac{p_{n-t}p'_t}{c-f_{n-t}-f'_t}=-\frac{u^2}{P_n};\quad \frac{q_{n-t}q'_t}{c-g_{n-t}-g'_t}=-\frac{v^2}{Q_n}\end{aligned}\right\}\qquad(14).$$

Hence we have

$$\frac{p_{n-t+1}q_t}{(c-f_{n-t+1}-g_t)^2}=\left(\frac{uv}{S_n}\right)^2\frac{1}{p_{n-t+1}q_t}=\frac{uv}{S^2_n}\cdot P_{n-t+1}Q_t;$$

from which we conclude that

$$\sum_{s=1}^{s=\infty}\sum_{t=1}^{t=\infty}\frac{p_s q_t}{(c-f_s-g_t)^2}=\sum_{n=1}^{n=\infty}\frac{uv}{S^2_n}\sum_{t=1}^{t=n}(P_{n-t+1}Q_t);$$

and, by using this and transformations similarly obtained for the other parts of the expression for F, we obtain

$$F=\sum_{n=1}^{n=\infty}\left\{\frac{uv}{S^2_n}\left[\sum_{t=1}^{t=n}(P_{n-t+1}Q_t)+\sum_{t=1}^{t=n-1}(S_{n-t}S_t)\right]\right.$$
$$\left.-\frac{u^2}{P^2_n}\left[\sum_{t=1}^{t=n-1}(P_{n-t}S_t)\right]-\frac{v^2}{Q^2_n}\left[\sum_{t=1}^{t=n-1}(Q_{n-t}S_t)\right]\right\}\qquad(15).$$

133. The quantities P_n, Q_n, S_n which occur in this expression, may be determined successively for successive values of n in the following manner:—By substituting, in (8), for p_n, p'_n, q_n, q'_n their values by (13), and eliminating f_n, f'_n, g_n, g'_n, we find

$$\left.\begin{array}{l} cP_n=aS_{n-1}+bS_n, \quad cQ_n=bS_{n-1}+aS_n \\ cS_{n-1}=bP_{n-1}+aP_n=aQ_{n-1}+bQ_n \end{array}\right\} \qquad (a);$$

from which we derive

$$\left.\begin{array}{l} P_{n+1}=\dfrac{c^2-a^2-b^2}{ab}P_n-P_{n-1} \\ Q_{n+1}=\dfrac{c^2-a^2-b^2}{ab}Q_n-Q_{n-1} \\ S_{n+1}=\dfrac{c^2-a^2-b^2}{ab}S_n-S_{n-1} \end{array}\right\} \qquad (b).$$

By giving n the values 1 and 2 in (13) and (8), we find

$$\left.\begin{array}{l} \left\{\begin{array}{l} P_1=\dfrac{1}{a}, \\ P_2=\dfrac{c^2-b^2}{a^2b}=\dfrac{c^2-a^2-b^2}{ab}P_1+\dfrac{1}{b}, \end{array}\right. \\ \left\{\begin{array}{l} Q_1=\dfrac{1}{b}, \\ Q_2=\dfrac{c^2-a^2}{ab^2}=\dfrac{c^2-a^2-b^2}{ab}Q_1+\dfrac{1}{a}, \end{array}\right. \\ \left\{\begin{array}{l} S_1=\dfrac{c}{ab}, \\ S_2=\dfrac{c^2-a^2-b^2}{ab}S_1 \end{array}\right. \end{array}\right\} \qquad (c).$$

By these equations we have directly the values of the first two terms of each of the sets of quantities P_1, P_2, P_3, etc., Q_1, Q_2, Q_3, etc., and S_1, S_2, S_3, etc.; and the others may be calculated successively by the preceding equations.

134. The polynomials which constitute the numerators of the successive terms of the second member of (15) may also be calculated successively, by means of equations obtained in the following manner. We have by (*c*), (*b*), and (*a*),

$$\begin{aligned} P_1Q_n+P_2Q_{n-1}+P_3Q_{n-2}+\text{etc.}&=\frac{1}{a}Q_n+\left(\frac{c^2-a^2-b^2}{ab}P_1+\frac{1}{b}\right)Q_{n-1} \\ &\quad+\left(\frac{c^2-a^2-b^2}{ab}P_2-P_1\right)Q_{n-2}+\text{etc.} \\ &=\frac{c}{ab}S_{n-1}+\frac{c^2-a^2-b^2}{ab}(P_1Q_{n-1}+P_2Q_{n-2}+\text{etc.})-(P_1Q_{n-2}+P_2Q_{n-3}+\text{etc.}); \end{aligned}$$

and similarly we find

$$\begin{aligned} &S_1S_{n-1}+S_2S_{n-2}+S_3S_{n-3}+\text{etc.} \\ &=\frac{c}{ab}S_{n-1}+\frac{c^2-a^2-b^2}{ab}(S_1S_{n-2}+S_2S_{n-3}+\text{etc.})-(S_1S_{n-3}+S_2S_{n-4}+\text{etc.}); \end{aligned}$$

$$S_1P_{n-1}+S_2P_{n-2}+S_3P_{n-3}+\text{etc.}$$
$$=\frac{c}{ab}P_{n-1}+\frac{c^2-a^2-b^2}{ab}(S_1P_{n-2}+S_2P_{n-3}+\text{etc.})-(S_1P_{n-3}+S_2P_{n-4}+\text{etc.})\,;$$

and

$$S_1Q_{n-1}+S_2Q_{n-2}+S_3Q_{n-3}+\text{etc.}$$
$$=\frac{c}{ab}Q_{n-1}+\frac{c^2-a^2-b^2}{ab}(S_1Q_{n-2}+S_2Q_{n-3}+\text{etc.})-(S_1Q_{n-3}+S_2Q_{n-4}+\text{etc.})$$

Hence, if we put

$$\left.\begin{aligned} &\sum_{t=1}^{t=n}(P_{n-t}Q_t)+\sum_{t=1}^{t=n-1}(S_{n-t}S_t)=2S'_n \\ &\sum_{t=1}^{t=n-1}(P_{n-t}S_t)=P'_n, \\ \text{and}\qquad &\sum_{t=1}^{t=n-1}(Q_{n-t}S_t)=Q_n \end{aligned}\right\}\qquad (e),$$

in terms of which notation the expression (15) for F becomes

$$\left.\begin{aligned} &F=2uv\left(\frac{S'_1}{S^2_1}+\frac{S'_2}{S^2_2}+\frac{S'_3}{S^2_3}+\text{etc.}\right) \\ &-\left\{u^2\left(\frac{P'_1}{P^2_1}+\frac{P'_2}{P^2_2}+\frac{P'_3}{P^2_3}+\text{etc.}\right)+v^2\left(\frac{Q'_1}{Q^2_1}+\frac{Q'_2}{Q^2_2}+\frac{Q'_3}{Q^2_3}+\text{etc.}\right)\right\} \end{aligned}\right\}(f);$$

we have

$$\left.\begin{aligned} S'_{n+1}&=\frac{c^2-a^2-b^2}{ab}S'_n-\left(S'_{n-1}-\frac{c}{ab}S_n\right) \\ P'_{n+1}&=\frac{c^2-a^2-b^2}{ab}P'_n-\left(P'_{n-1}-\frac{c}{ab}P_n\right) \\ Q'_{n+1}&=\frac{c^2-a^2-b^2}{ab}Q'_n-\left(Q'_{n-1}-\frac{c}{ab}Q_n\right) \end{aligned}\right\}\qquad (g).$$

Also we have directly from (*e*) and (*c*),

$$\left.\begin{aligned} &S'_1=\tfrac{1}{2}\frac{1}{ab}, &&S'_2=\tfrac{1}{2}\frac{3c^2-a^2-b^2}{a^2b^2}\,. \\ &P'_1=0, &&P'_2=\frac{c}{a^2b} \\ &Q'_1=0, &&Q'_2=\frac{c}{ab^2} \end{aligned}\right\}\qquad (h).$$

135. These equations enable us to calculate successively the values of S'_1, S'_2, S'_3, etc., P'_1, P'_2, P'_3, etc., and Q'_1, Q'_2, Q'_3, etc., after the values of S_1, S_2, etc., P_1, P_2, etc., and Q_1, Q_2, etc., have been found.

136. The solution of (*b*) as equations of finite differences with

reference to n, and the determination of the arbitrary constants of integration by (c), leads to general expressions for S_n, P_n, and Q_n; and by using these in (g), integrating the equations so obtained, and determining the arbitrary constants by means of (h), general expressions for S'_n, P'_n, and Q'_n are obtained. The expression for F may therefore be put in the form of an infinite series, with a finite expression for the general term. Further, the value of this series may be expressed, by means of analysis similar to that which Poisson has used for similar purposes, in terms of a definite integral. I do not, however, in the present communication give any of this analysis, except for the case of two spheres in contact which is discussed below, because, except for cases in which the spheres are very near one another, the series for F is rapidly convergent, and the terms of it may be successively calculated with great ease, by regular arithmetical processes, for any set of values of c, a, and b, by using first the equations (c), to calculate S_1, S_2, P_1, P_2, Q_1, Q_2; then (b) with the values 2, 3, etc., successively substituted for n, to calculate S_3, S_4, etc., and P_3, P_4, etc., and Q_3, Q_4, etc.; then (h) and (g) to calculate by a similar succession of processes, the values of S'_1, S'_2, S'_3, etc., P'_1, P'_2, P'_3, etc., and Q'_1, Q'_2, Q'_3, etc.

137. The following is the method, alluded to above, by which I first arrived at the solution of this problem in the year 1845.

138. The "mechanical value" of a distribution of electricity on a group of insulated conductors, may be easily shown to be equal to half the sum of the products obtained by multiplying the quantity of electricity on each conductor into the potential within it.* Hence, if D and E denote the quantities of electricity on the two spheres in the present case, and if W denote the mechanical value of the distribution of electricity on them, we have

$$W=\tfrac{1}{2}(Du+Ev).$$

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* This proposition occurred to me in thinking over the demonstration which Gauss gave of the theorem *that a given quantity of matter may be distributed in one and only one way over a given surface so as to produce a given potential at every point of the surface*, and considering the mechanical signification of the function on the rendering of which a minimum that demonstration is founded. It was published, I believe, by Helmholtz in 1847, in his treatise *Ueber die Erhaltung der Kraft*, by the translation of which, in the last number of the *New Scientific Memoirs*, a great benefit has been conferred on the British scientific public.

Now if the two spheres, kept insulated, be pushed towards one another, so as to diminish the distance between their centres from c to $c-dc$, the quantity of work that will have to be spent will be $F.dc$, since F denotes the repulsive force against which this relative motion is affected. But the mechanical value of the distribution in the altered circumstances must be increased by an amount equal to the work spent in producing no other effect but this alteration. Hence $F.dc = -dW$, and therefore

$$F = -\frac{1}{2}\frac{d(Du+Ev)}{dc} \qquad (16),$$

where u and v are to be considered as varying with c, and D and E as constants. Now, according to the notation expressed in (13), we have

$$\left.\begin{array}{r} \left(\frac{1}{P_1}+\frac{1}{P_2}+\text{etc.}\right)u-\left(\frac{1}{S_1}+\frac{1}{S_2}+\text{etc.}\right)v=D \\ -\left(\frac{1}{S_1}+\frac{1}{S_2}+\text{etc.}\right)u+\left(\frac{1}{Q_1}+\frac{1}{Q_2}+\text{etc.}\right)v=E \end{array}\right\} \qquad (17).$$

Determining $\frac{du}{dc}$ and $\frac{dv}{dc}$ by the differentiation of these equations, and using the results in (16), we find

$$F=\frac{1}{2}\left\{u^2\frac{d}{dc}\left(\frac{1}{P_1}+\frac{1}{P_2}+\text{etc.}\right)-2uv\frac{d}{dc}\left(\frac{1}{S_1}+\frac{1}{S_2}+\text{etc.}\right)\right.$$
$$\left.+v^2\frac{d}{dc}\left(\frac{1}{Q_1}+\frac{1}{Q_2}+\text{etc.}\right)\right\} \qquad (18).$$

This expression agrees perfectly with (f), given above; since, by differentiating the equations (b) and (c) with reference to c, we find that the quantities denoted above by S'_1, S'_2, S'_3, etc., P'_1, P'_2, P'_3, etc., Q'_1, Q'_2, Q'_3, etc., and expressed by the equations (g) and (h), are equal respectively to

$$\frac{1}{2}\frac{dS_1}{dc},\ \frac{1}{2}\frac{dS_2}{dc},\ \frac{1}{2}\frac{dS_3}{dc},\ \text{etc.},\ \frac{1}{2}\frac{dP_1}{dc},\ \frac{1}{2}\frac{dP_2}{dc},\ \frac{1}{2}\frac{dP_3}{dc},\ \text{etc.},$$
$$\frac{1}{2}\frac{dQ_1}{dc},\ \frac{1}{2}\frac{dQ_2}{dc},\ \frac{1}{2}\frac{dQ_3}{dc},\ \text{etc.}$$

139. The series (f) or (18) for F becomes divergent for the case of two spheres in contact, but the doubly infinite series from which this was derived in the first of the two investigations given above, is convergent when the terms are properly grouped together; and its sum may be expressed by means of a definite integral in the following manner:—

140. Since the two spheres are in contact, the potentials within them must be equal, that is, we must have $u=v$. For the sake of simplicity, let us suppose the radii of the two spheres to be equal, and let each be taken as unity. Then we shall have $a=b=1$, and $c=2$; and the terms of doubly infinite series (9) in this case are easily expressed,* in very simple forms, by equations (8). Thus we find

$$F=v^2\times\left\{\begin{array}{l}\frac{1}{2^2}-\frac{1.2}{3^2}+\frac{1.3}{4^2}-\frac{1.4}{5^2}+\frac{1.5}{6^2}-\text{etc.}\\ \qquad-\frac{2.1}{3^2}+\frac{2.2}{4^2}-\frac{2.3}{5^2}+\frac{2.4}{6^2}-\text{etc.}\\ \qquad\qquad+\frac{3.1}{4^2}-\frac{3.2}{5^2}+\frac{3.3}{6^2}-\text{etc.}\\ \qquad\qquad\qquad-\frac{4.1}{5^2}+\frac{4.2}{6^2}-\text{etc.}\\ \qquad\qquad\qquad\qquad+\frac{5.1}{6^2}-\text{etc.}\end{array}\right\}\qquad(k).$$

If we add the terms in the vertical columns, we find

$$F=v^2\times\tfrac{1}{6}\left(\frac{1.2.3}{2^2}-\frac{2.3.4}{3^2}+\frac{3.4.5}{4^2}-\text{etc.}\right),$$

which is a diverging series, and is the same as we should have found by using the form (f) or (18). But if we add the terms in the horizontal lines, we find the following convergent series for F:—

$$F=v^2\left\{\int_0^1\frac{\log\frac{1}{\theta}\cdot\theta d\theta}{(1+\theta)^2}-2\int_0^1\frac{\log\frac{1}{\theta}\cdot\theta^2 d\theta}{(1+\theta)^2}+3\int_0^1\frac{\log\frac{1}{\theta}\cdot\theta^3 d\theta}{(1+\theta)^2}-\text{etc.}\right\}$$

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* From equations (8) we find, in this case,

$$f'_n=g'_n=\frac{2n-1}{2n}\,;\quad f_n=g_n=\frac{2n-2}{2n-1}$$

$$p'_n=q'_n=-\frac{v}{2n}\,;\quad p_n=q_n=\frac{v}{2n-1}\,.$$

Hence
$$\frac{p_s q_t}{(c-f_s-g_t)^2}=\frac{(2s-1)(2t-1)}{\{2(s+t)-2\}^2}$$

$$\frac{p'_s q'_t}{(c-f'_s-g'_t)^2}=\frac{2s.2t}{\{2(s+t)\}^2}$$

$$\frac{p_s p'_t}{(c-f_s-f'_t)^2}=\frac{q_s q'_t}{(c-g_s-g'_t)}=-\frac{2t(2s-1)}{\{2(s+t)-1\}^2}\,;$$

and then, by (9), we obtain the expression for F in this particular case, given in the text.

Hence, since $(1+\theta)^{-2}=1-2\theta+3\theta^2-$etc., we have

$$F=v^2\int_0^1\frac{\log\frac{1}{\theta}\cdot\theta d\theta}{(1+\theta)^4}\qquad(l);$$

or, by actual integration,

$$F=v^2\left[\frac{\log\frac{1}{\theta}}{(1+\theta)^3}\,\frac{3\theta^2+\theta^3}{6}+\tfrac{1}{6}\log(1+\theta)-\tfrac{1}{6}\frac{\theta}{(1+\theta)^3}\right]_0^1$$
$$=v^2.\tfrac{1}{6}\times(\log 2-\tfrac{1}{4})=v^2.\tfrac{1}{6}\times(\cdot 69315-\cdot 25)$$
$$=v^2\times\cdot 073858.$$

The quantity of electricity on each sphere being equal to the sum of the masses of the imaginary series of points within it, is, according to the formulæ for p_1, q'_1, p_2, q'_2, etc.,

$$v(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\text{etc.}),\text{ or } v\log 2.$$

Hence we have the following expression for the repulsion between the two spheres, in terms of Q the quantity of electricity on each,

$$F=Q^2.\frac{\frac{1}{6}\times(\log 2-\frac{1}{4})}{(\log 2)^2}.$$

141. If x denote the distance at which two electrical points, containing quantities equal to the quantities on the two spheres, must be placed so as to repel one another with a force equal to the actual force of repulsion between the spheres, we have

$$\frac{(v.\log 2)^2}{x^2}=F.$$

Using the value for F found above, we obtain

$$x=\frac{\log 2}{\sqrt{\{\frac{1}{6}\times(\log 2-\frac{1}{4})\}}}=2\cdot 550.$$

If the electrical distribution on each surface were uniform, this distance would be equal to 2, the distance between the centres of the spheres; but it exceeds this amount, to the extent shown by the preceding result, because in reality the electrical density on each conductor increases gradually from the point of contact to the remotest points of the two surfaces.

P.S.—The calculation by the method shown in the preceding paper, of the various quantities required for determining the force between two *spheres of equal radii* (each unity), insulated with their centres at distances 2·1, 2·2, 2·3, etc., up to 4, has been undertaken, and is now nearly complete.

GLASGOW COLLEGE, *March* 21, 1853.

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142. The following numerical results have been calculated (by means of the formulæ established above) for application to the theory of a new electrometer which I have recently had constructed to determine electrical potentials in absolute measure, from the repulsions of uninsulated balls in the interior of a hollow insulated and electrified conductor, by means of a bifilar or torsion balance bearing a vertical shaft which passes through a small aperture to the outside of the conductor:—

TABLE I.—*Showing the Quantities of Electricity on two equal Spherical Conductors, of radius* r, *and the mutual force between them, when charged to potentials* u *and* v *respectively.*

Col. 1.	Cols. 2 and 3.		Cols. 4 and 5.		Col. 6.
Distance from centre to centre ÷r.	For determining the quantities of electricity, $D=(Iu-Jv)r$ $E=(Iv-Ju)r$.		For determining the mutual force, $F=2Buv-A(u^2+v^2)$; being repulsion when positive, and attraction when negative.		Ratio of the potentials when there is neither attraction nor repulsion, $\rho=\frac{B}{A}-\left(\frac{B^2}{A^2}-1\right)^{\frac{1}{2}}$ $\frac{1}{\rho}=\frac{B}{A}+\left(\frac{B^2}{A^2}-1\right)^{\frac{1}{2}}$.
a.	I.	J.	A.	B.	ρ.
2·0	$J+{}$·698147	∞	∞	$A+\frac{1}{2}\times$·078858	$1-\sqrt{\frac{·078858}{A}}$
2·1	1·58396	·88175	1·13844	1·17439	·77828
2·2	1·43131	·72378	·52852	·56350	·69637
2·3	1·34827	·63395	·32917	·36357	·63553
2·4	1·29316	·57202	·23159	·26464	·58975
2·5	1·25324	·52537	·17432	·20630	·55888
2·6	1·22218	·48819	·13696	·16787	·51699
2·7	1·19755	·45746	·11082	·14090	·47805
2·8	1·17738	·43140	·09174	·12078	·46049
2·9	1·16056	·40886	·07720	·10526	·43667
3·0	1·14629	·38908	·06592	·09299	·41567
3·1	1·13404	·37151	·05693	·08304	·39672
3·2	1·12340	·35571	·04963	·07481	·37947
3·3	1·11410	·34150	·04363	·06791	·36376
3·4	1·10588	·32852	·03863	·06208	·34939
3·5	1·09859	·31663	·03441	·05697	·33615
3·6	1·09208	·30569	·03084	·05257	·32418
3·7	1·08623	·29557	·02775	·04872	·31263
3·8	1·08095	·28617	·02509	·04531	·30211
3·9	1·07617	·27742	·02278	·04229	·29238
4·0	1·07182	·26924	·02075	·03958	·28318

TABLE II.—*Showing the Potentials in two equal Spherical Conductors, and the mutual force between them, when charged with quantities* D *and* E *of electricity respectively.*

Col. 1.	Cols. 2 and 3.		Cols. 4 and 5.		Col. 6.
Distance from centre to centre $=ar$.	For determining the potentials, $u=\left(\frac{I}{I^2-J^2}\cdot D+\frac{J}{I^2-J^2}\cdot E\right)\frac{1}{r}$ $v=\left(\frac{I}{I^2-J^2}\cdot E+\frac{J}{I^2-J^2}\cdot D\right)\frac{1}{r}$.		For determining the mutual force, $F=\left\{2\beta DE-\alpha(D^2+E^2)\right\}\frac{1}{r^2}$, where $\alpha=\frac{A(I^2+J^2)-2BIJ}{(I^2-J^2)^2}$ $\beta=\frac{B(I^2+J^2)-2AIJ}{(I^2-J^2)^2}$.		Ratio of the quantities when there is neither attraction nor repulsion, $\theta=\frac{I\rho-J}{I-J\rho}$.
a.	$\frac{I}{I^2-J^2}$.	$\frac{J}{I^2-J^2}$.	α.	β.	θ.
2·0	$\frac{1}{2}\times\frac{1}{\cdot693147}$	$\frac{1}{2}\times\frac{1}{\cdot693147}$	∞	$\alpha+\frac{1}{2}\times\cdot153726$	$1-\sqrt{\frac{\cdot153726}{\alpha}}$
2·1	·91482	·50926	·15375	·22668	·39102
2·2	·93869	·47467	·08268	·15251	·29435
2·3	·95220	·44782	·05444	·12186	·23530
2·4	·96142	·42528	·03955	·10309	·19944
2·5	·96829	·40599	·02997	·09038	·16908
2·6	·97354	·38888	·02342	·08078	·14476
2·7	·97771	·37348	·01849	·07341	·12786
2·8	·98105	·35946	·01500	·06710	·11318
2·9	·98376	·34658	·01222	·06186	·09971
3·0	·98598	·33467	·01010	·05731	·08877
3·1	·98782	·32361	·00842	·05333	·07944
3·2	·98934	·31327	·00708	·04981	·07139
3·3	·99067	·30366	·00599	·04666	·06442
3·4	·99178	·29462	·00510	·04382	·05839
3·5	·99272	·28612	·00437	·04126	·05298
3·6	·99351	·27810	·00378	·03891	·04868
3·7	·99423	·27054	·00326	·03679	·04349
3·8	·99484	·26338	·00283	·03484	·04061
3·9	·99537	·25659	·00247	·03305	·03736
4·0	·99583	·25015	·00216	·03139	·03444

VII. ON THE ATTRACTIONS OF CONDUCTING AND NON-CONDUCTING ELECTRIFIED BODIES.

[From the *Cambridge Mathematical Journal*, May 1843.]

144. In measuring the action exerted upon an electrified body, by a quantity of free electricity distributed in any manner over another body, the methods followed in the cases in which the attracted body is conducting and non-conducting are different. Now, the only difference between the state of a conducting body and that of a non-conducting body is, that the electricity is held upon a conducting body by the pressure of the atmosphere (to a certain extent at least), while on a non-conducting body it is held by the *friction* of the particles of the body.

145. To find the attraction of an electrical mass E, on a non-conducting electrified body A, the obvious way is to proceed as in ordinary cases of attraction, considering the electricity on A as the attracted mass.

In finding the action on a conducting body A, the method followed is to consider its electricity as exerting no pressure upon the particles of the body, but disturbing its equilibrium, by making the pressure of the air unequal at different parts of its surface. These two methods of measuring the action of E on A *should* obviously lead to the same result, since the action must be the same, whether A be conducting or non-conducting, the distribution remaining the same. It is the object of the following paper to show that they *do* lead to the same result.

146. We must first find the pressure of an element of the electricity of A, on the atmosphere.

Let ds be the area of the element, and ρds its electrical mass. Let ds form part of another element σ, indefinitely larger than ds in every direction, but so small that it may be considered as plane. Now, if $\rho\sigma$ be a material plane, it can exercise no attraction on ρds, in a direction perpendicular to the plane, and it may be readily shown that this is also true if $\rho\sigma$ be a plate of matter of different densities, arranged in parallel planes, the thickness being either finite or indefinitely small, and the law of density being any whatever.

147. Hence, the force acting on ρds is due to the repulsion of all the electrical mass, except σ; and, since the electricity on A is in equilibrium under the influence of E, the repulsion acts along the normal through ds, and is in magnitude $2\pi\rho^2 ds$ (see I. above, § 7), which is therefore the pressure of ds on the air. Hence, if p be the barometric pressure of the atmosphere, the pressure on ds, perpendicular to the surface, is

$$(p-2\pi\rho^2)ds.$$

Hence, if X be the whole pressure on A, resolved along a fixed line $X'X$, and if ν be the angle which the normal through ds makes with this line, we have

$$X=-\iint(p-2\pi\rho^2)\cos\nu\, ds,$$

the integrals being extended over the surface of A. Now,

$$\iint p\cos\nu\, ds=0,$$

since the pressure of the atmosphere does not disturb the equilibrium of A. Hence, we have

$$X=2\pi\iint\rho^2\cos\nu\, ds \qquad (a),$$

which is the expression for the attraction on a conducting body A, either separate from the body on which E is distributed, or connected with it.

148. To show that this is identical with the expression for the attraction of E on the electricity of A, let $R\rho ds$ and $R'\rho ds$ be the components of the repulsion on ρds, which are due to E, and to the electricity of A; and let a, a' be the angles which their directions make with XX'. Then we shall have

$$2\pi\rho\cos\nu=R\cos a+R'\cos a';$$

therefore
$$X=\iint(R\cos a+R'\cos a')\,ds.$$

Now, $\iint R'\cos a'\,ds$ is the attraction of the electricity of A on itself in the direction XX', and is therefore $=0$. Hence,

$$X=\iint R\cos a\,ds \qquad (b).$$

But this expression for X is the attraction of E on the electricity of A: [also, the *moment* round OX is the same for the diminution of air pressure as for the attraction of E on the electricity of A:] and hence the two methods of measuring the action lead to the same result.

VIII. DEMONSTRATION OF A FUNDAMENTAL PROPOSITION IN THE MECHANICAL THEORY OF ELECTRICITY.

[From the *Cambridge Mathematical Journal*, Feb. 1845.]

149. If a material point be in a position of equilibrium when under the influence of any number of masses attracting it or repelling it with forces which are inversely proportional to the square of the distance, the equilibrium will be unstable.*

The first thing to be proved is, that if the material point receive a slight displacement, there will in general be a moving force called into action.

150. Let O be the position of equilibrium; P any adjacent point; V the potential of the influencing masses, μ, at P, which point we suppose not to be contained within any portion of μ; U the value of V at O. Now it is shown by Gauss, in his Mémoire on General Theorems in Attraction, that V cannot have the constant value U through any finite volume, however small, adjacent to O, without having it for every point external to μ. But this is impossible, as may be shown in the following manner.

Let σ be a closed surface containing within it a quantity of matter, μ_1, consisting of any number of detached portions of μ, or of the whole of μ, if μ be a continuous mass. Let $d\sigma$ be an element of σ, and P the force due to the total action of μ, resolved in a direction perpendicular to $d\sigma$, which may be considered positive when directed towards the space within σ. Then, by a theorem demonstrated in this *Journal* (see XII. below, § 200), we have $\iint P d\sigma = 4\pi\mu_1$

the integrations being extended over the whole of σ. Hence P cannot be $= 0$ for every point of the surface σ, and therefore V cannot be constant for all the space exterior to μ.

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* This theorem was first given by Mr. Earnshaw, in his Memoir on Molecular Forces, read at the Cambridge Philosophical Society, March 18, 1839. See vol. VII. of the *Transactions*.

Hence V cannot have the constant value U for every point of any finite volume, however small, adjacent to O.

151. Now let a sphere S be described round O as centre, with any radius a, sufficiently small that no portion of μ shall be included, and let P be any point of the surface S, and ds an element of the surface at P.

In the equations (3) and (4) of the article already referred to (XII. below, § 199), let the sphere S be the surface there considered; let $v=V$, and $v_1=\frac{1}{r}$, if $OP=r$.

Hence $P_1=\frac{1}{a^2}$ and $v_1=\frac{1}{a}$, at every point of S;

$$m=\mu,\quad m_1=1,\quad \iiint v\,dm_1=U.$$

Also $$\iint v_1 P\,ds=\frac{1}{a}\iint P\,ds=0,$$

and $\iiint v_1 dm=0$, since S does not contain any of the matter μ. We have therefore, by comparing (3) and (4) of § 199,

$$0=4\pi U-\frac{1}{a^2}\iint V\,ds.$$

Therefore $$\iint V\,ds=4\pi a^2 U,$$

which shows that the mean value for the surface of a sphere, of the potential of any external masses, is equal to the value at the centre. Let $$V=U+u.$$

Therefore $$\iint u\,ds=0.$$

152. Now, as has already been shown, u cannot be $=0$ for every point P adjacent to O, and therefore if the sphere pass through a point P' where u is negative, there must also be a point P'' in the surface, for which u is positive. But if we assume the potential of an attracting particle to be positive, the direction of the resultant force, resolved along any straight line, will be that in which V increases. Hence there will be a force towards O, for points displaced along OP', and from O, for points displaced along OP''. Hence if M, the material point in equilibrium at O, be displaced along OP'', the moving force generated will tend to remove it further from O, which is therefore an unstable position.

153. As an application of this theorem, let us consider the case of any number of material points repelling one another ac-

cording to the inverse square of the distance, and contained in the interior of a rigid closed envelope. Let the system be in equilibrium when acted upon by attracting or repelling masses distributed in any manner without the envelope.

It will generally be possible that there may be a position or positions of equilibrium, in which at least some of the particles are not in contact with the surface. If now we suppose all the particles fixed except one, not in contact with the surface, the equilibrium of this particle is, as has been shown, unstable. Hence, generally, the equilibrium of the system is unstable if any of the particles be not in contact with the surface, and therefore in nature the particles cannot remain in such a position. There must, however, be some stable position or positions in which the particles can rest, but in such, all the particles must be in contact with the surface of the envelope. The sole condition of equilibrium in this case will be that the resultant force on each particle shall be in the direction of a normal to the surface, and directed towards the exterior space. If the number of particles be infinite, and there be one position in which the whole surface is covered, there can be no other in which this is the case, as is shown in the paper in this *Journal* already quoted (XII. below, § 204); and it is also readily seen that this position will be stable, and that no other in which the surface is not entirely covered can be stable. In this case the particles will be distributed according to the law of the intensity of electricity on the surface, the space within being conducting matter, and the masses without being any electrified bodies. If a mechanical theory be adopted, *electricity* will actually be a number of material points without weight, which repel one another according to the inverse square of the distance. Thus the result we have arrived at is, that there can be permanently no free electricity in the interior of a conducting body under any circumstances whatever.

154. If, as may happen through the influence of the exterior masses, there cannot be a position of equilibrium of the particles covering the whole surface, there will be a permanent distribution, in which part of the surface is uncovered. This, however, is never the case with electricity, as a certain quantity of latent electricity is then decomposed, so that the whole

surface is covered with electricity, either positive or negative. All the above reasoning would still apply, if we considered the masses of some points to be negative, and of some positive, and the force between any two to be a repulsion equal to the product of their masses divided by the square of their distance.

155. Since every particle is on the surface, the whole *medium* (if it can be properly so called), will be an indefinitely thin stratum, the thickness being in fact the ultimate breadth of an atom or material point. If we suppose these atoms to be merely centres of force, the thickness will therefore be absolutely nothing, and thus the *fluid* will be absolutely compressible and inelastic. Any thickness which the stratum can have must depend on a force of elasticity, or on a force generated by the contact of material points, and in either case will therefore require * an ultimate law of repulsion more intense than that of the inverse square,† when the distance is very small, and we therefore conclude that this cannot be the ultimate law of repulsion in any elastic fluid. As, however, all experiments yet made serve to confirm the fact that there is no electricity in the interior of conducting bodies, or that the stratum has absolutely no thickness, we conclude that there is no elasticity in the assumed electric fluid, and thus the law of force, deduced independently by direct experiments, is confirmed.

ST. PETER'S COLLEGE, *Jan.* 16, 1845.

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* [*Note added Jan.* 1869.—This was written without knowledge of Davy's "repulsive motion," and without the slightest idea that elasticity of every kind is most probably a result of motion. The conclusions of the text are, however, not affected by these views.]

† This agrees with a result of Mr. Earnshaw.

IX. NOTE ON INDUCED MAGNETISM IN A PLATE.

[From the *Cambridge and Dublin Mathematical Journal*, Nov. 1845.]

156. If a plate of soft iron be submitted to the action of a magnet of any kind, it immediately becomes magnetized "by induction;" and the effects of this are exhibited in the attraction or repulsion it exercises upon small magnetic bodies in its neighbourhood. The determination of these effects, from the elementary laws of magnetic induction, is a problem of considerable practical interest. In the case of a plate bounded by infinite parallel planes, I have succeeded in obtaining a complete solution of a very simple nature, by means of a principle which will be developed in a future paper (see above, §§ 127, 107, 108, 44). The object of the present note is to compare this solution with a formula given by Green in his *Essay on Electricity and Magnetism*, as an approximate result, but which appears to be inadmissible.

157. Let the influencing magnet, which may be of any form and size, and magnetized in any manner, be denoted by Q; and let us suppose it to be held *behind* the plate of soft iron. The solution which I have obtained enables us to find the total magnetic action on a point, P, situated in any position, either within or without the plate; but at present I shall only state the result when P is *before* the plate. In this case the actual magnetic effect on P may be produced by supposing Q and the plate to be removed, and a certain imaginary series of magnets Q', Q_1, Q_2, etc., to be substituted, the system being constructed thus. Each of the imaginary magnets is equal and similar to Q, and similarly magnetized; Q' occupies the place of Q, and the others are similarly placed behind it, along a line perpendicular to the plate, the distance between corresponding points of each consecutive pair being equal to twice the thickness of the plate. The intensities of the successive magnets decrease in a geometrical progression, of which the common ratio is m^2 (a

quantity measuring (§ 45) the inductive capacity for magnetism of the plate), commencing with that of Q', which is equal to $1-m^2$, if the intensity of Q be unity. It is hardly necessary to point out the analogy between this and the corresponding result in optics, in which the illumination produced through a plate of glass, by a candle, is found to be due to the candle itself, with diminished brightness, and to a row of images behind it, with intensities decreasing in a geometrical progression, which arise from successive internal reflections.

158. If the iron plate be infinitely thin, all the *images*, Q_1, Q_2, etc., will coincide with Q'; and, since the sum of their intensities is unity, the total effect will be the same as that of Q, which will therefore be unaffected by the interposition of the screen. The same will be the case if the distance of Q be infinitely great, and the thickness of the screen finite; but in this case, at least as far as the present result can show us, the dimensions of the planes which bound the plate must be infinitely great compared with the distance of Q.

159. The result which I have stated is applicable also to the imaginary case in which, instead of being a magnet, Q is a mass of positive or negative magnetism.* Thus, let Q be a unit of positive magnetism collected in a point, which case is investigated by Green. To express the action analytically, let Q be taken as origin of co-ordinates, a line perpendicular to the plate as axis of x, and the plane through this line, and P, as plane of (x, y). Then denoting by a the thickness of the plate, and considering Q as a positive unit of matter, we shall have, for the total potential at P, due to Q and the plate,

$$F=(1-m^2)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}}}+\frac{m^2}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}}+\frac{m^4}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}}+\text{etc.}\right\} \quad (1).$$

160. For all magnetic bodies m is between 0 and 1, the former limit being its value when the inductive capacity for

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* This expression does not imply any hypothesis of a magnetic matter or of a fluid or fluids, but it is merely used for brevity in consequence of the principle established by Coulomb, Poisson, and Ampère, that the action of a magnetized body of any kind, or of a collection of electric "closed currents," may always be represented by an imaginary positive and negative distribution of matter, of which the whole mass is algebraically nothing. By an element of positive or negative magnitude, we merely mean a portion of this imagined matter.

magnetism is nothing, and the latter being never attained, though it is approached in such bodies as iron, of which the inductive capacity is great. In the extreme case of $m=1$, the laws of induction in a magnetic body degenerate into those of electrical equilibrium on the surface of a conductor of electricity. If in the expression for F we put $m=1$, one of the factors vanishes and the other becomes infinite, but the ultimate value of the product is nothing, which shows that the effect of the plate is to destroy all action behind it. This we know to be the case when an infinite conducting screen of any form is placed before an electrified body.

161. In the case when the plate is of iron, the value of m is nearly unity. Hence, as the series is multiplied by $1-m^2$, it might be imagined that, if we "neglect small quantities of the order $(1-g)$ compared with those which are retained," ($1-g$ being, in Green's notation, a quantity of the same order as $1-m$), an approximate result would be obtained by putting $m=1$ in the successive terms of the series within the vinculum. And it is thus that Green, having, in the investigation, neglected quantities multiplied by $(1-g)^2$, arrives at the result,

$$F=\frac{4(1-g)}{3}\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}}}+\frac{1}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}}+\frac{1}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}}+\text{etc.}\right\}$$

As, however, this series has an infinite sum, it is clear that no value of m can be sufficiently near to unity to render the approximation admissible. If instead of Q we were to substitute a magnet, or any collection of positive and negative particles, such that the sum of the masses is zero, the series for the potential, deduced from Green's expression, would converge: and the same remark is applicable to the series which would be found for the *attraction* of the system on a point beyond the screen, even when Q is a positive point, by differentiating the expression for F. Notwithstanding this, the approximation is still inadmissible; since, if we expand the rigorous expression in either case in ascending powers $(1-m)$, we find that, though the first term is finite, the co-efficients of all the terms which follow it are infinite.

162. Although the method by which I obtained the rigorous

solution is quite distinct from that followed by Green, being independent of any mathematical process, it may be satisfactory to show that the result can be deduced from his own analysis, and even with greater ease than his solution is obtained after making unnecessary approximation.

By a very remarkable investigation, in which he extends Laplace's well-known analysis for spherical co-ordinates to the case when the radius of the sphere becomes infinite, Green arrives (*Essay on Electricity*, p. 64) at the following expression for the total potential at P, due to the positive unit of matter Q, and to the interposed plate, before making any approximation:—

$$F=\frac{8}{\pi}(1-g)\,(1+2g)\int_0^\infty\frac{d\gamma\,\epsilon^{-\gamma x}}{(2+g)^2-9g^2\epsilon^{-2\gamma a}}\int_0^1\frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}\cos(\beta\gamma y).$$

Let $m=\frac{3g}{2+g}$. Then we have, by expansion, and by changing the order of the integration,

$$F=\frac{2}{\pi}(1-m^2)\int_0^1\frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$
$$\int_0^\infty d\gamma.\epsilon^{-\gamma x}(1+m^2\epsilon^{-2\gamma a}+m^4\epsilon^{-4\gamma a}+\text{etc.})\cos(\beta\gamma y)$$
$$=\frac{2}{\pi}(1-m^2)\int_0^1\frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$
$$\left\{\frac{x}{x^2+\beta^2y^2}+\frac{m^2(x+2a)}{(x+2a)^2+\beta^2y^2}+\frac{m^4(x+4a)}{(x+4a)^2+\beta^2y^2}+\text{etc.}\right\}.$$
$$=\frac{2}{\pi}(1-m^2)\Sigma\int_0^{\frac{1}{2}\pi}\frac{m^{2i}x_i\,d\theta}{x_i^2+y^2\sin^2\theta},\ \text{where } x_i=x+2ia,$$
$$=(1-m^2)\Sigma\frac{m^{2i}}{(x_i^2+y^2)^{\frac{1}{2}}},$$

which agrees with the expression given above.

ST. PETER'S COLLEGE, *Oct.* 14*th*, 1845.

X. SUR UNE PROPRIÉTÉ DE LA COUCHE ÉLECTRIQUE EN ÉQUILIBRE À LA SURFACE D'UN CORPS CONDUCTEUR.

Par *M*. J. Liouville.

[From the *Cambridge and Dublin Mathematical Journal*, Nov. 1846.]

163. La méthode la plus générale que l'on connaisse pour former des couches électriques, en équilibre à la surface de corps conducteurs, consiste à considérer une masse M; et le potentiel,

$$V = \iiint \frac{f(x', y', z')\, dx'dy'dz'}{\Delta},$$

de cette masse, par rapport à un point quelconque (x, y, z), dont la distance au point (x', y', z'), ou à l'élément

$$f(x', y', z')dx'dy'dz',$$

est désignée par Δ. Prenons ensuite une surface de niveau ou d'équilibre relativement à l'attraction de la masse M, et qui entoure cette masse, c'est à dire prenons une surface fermée (A), contenant la masse M dans son intérieur, et pour tous les points de laquelle V conserve une valeur constante. En fin soit $\frac{dV}{ds}ds$ la variation infiniment petite que V éprouve lorsqu'on passe d'un point de cette surface à un point extérieur infiniment voisin situé sur la normale à une distance ds. C'est la dérivée $\frac{dV}{ds}$, multipliée si l'on veut par une constante, qui réglera la loi des densités de l'électricité en équilibre sur un corps conducteur terminé par la surface (A). Plusieurs géomètres sont parvenus, chacun de leur côté, à ce beau théorème; mais c'est George Green qui l'a, je crois, donné le premier dans un excellent mémoire publié en 1828, sous ce titre: *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Je me propose de montrer que la couche électrique en équilibre ainsi obtenue a précisément le même centre de gravité que la masse M.

164. Plaçons l'origine des coordonnées x, y, z, au centre de

gravité de la masse M; et désignons par x_1 une quelconque des coordonnées du centre de gravité de la couche électrique, laquelle sera fournie par la formule

$$x_1 \int\int \frac{dV}{ds} d\omega = \int\int x \frac{dV}{ds} d\omega,$$

où les intégrations s'appliquent à la surface (A) dont l'élément est représenté par $d\omega$. Il s'agit de prouver que $x_1 = 0$.

D'après l'expression de V, on a

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z), \text{ ou } = 0,$$

suivant que le point (x, y, z) appartient ou non à la masse M.

Pour plus de simplicité, écrivons toujours

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z),$$

en regardant la fonction $f(x, y, z)$ comme nulle hors de la masse M; et combinons cette équation avec cette autre de forme analogue

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0,$$

où nous supposons que U est une fonction de x, y, z, qui reste finie et continue ainsi que ses dérivées dans tout l'espace intérieur à (A). Nous aurons

$$V\frac{d^2U}{dx^2} - U\frac{d^2V}{dx^2} + V\frac{d^2U}{dy^2} - U\frac{d^2V}{dy^2} + V\frac{d^2U}{dz^2} - U\frac{d^2V}{dz^2} = 4\pi Uf(x, y, z).$$

Multiplions par $dx\,dy\,dz$, et intégrons dans tout l'espace intérieur à (A). En conservant à ds et à $d\omega$ la même signification que ci-dessus, on trouve, après des transformations bien connues :

$$\iint V\frac{dU}{ds}d\omega - \iint U\frac{dV}{ds}d\omega = 4\pi\iiint Uf(x, y, z)dxdydz.$$

Mais l'équation en U est satisfaite par $U = x$; nous avons donc :

$$\iint V\frac{dx}{ds}d\omega - \iint x\frac{dV}{ds}d\omega = 4\pi\iiint xf(x, y, z)dxdydz.$$

L'intégrale triple du second membre, divisée par M, donne l'abscisse du centre de gravité de la masse M. Ce centre étant à l'origine des coordonnées, l'intégrale dont nous parlons est nulle. Je vais prouver que l'intégrale $\iint V\frac{dx}{ds}\,d\omega$ l'est aussi. D'abord on peut faire sortir V du signe $\int$, puisque, sur la surface (A), V est constant. Observons ensuite que $\frac{dx}{ds}$ a pour valeur le cosinus de l'angle a que la normale ds fait avec l'axe des x. Notre intégrale deviendra donc: $V\iint\cos a\,d\omega$. Or l'intégrale $\iint\cos a\,d\omega$ est nulle, d'après un théorème connu, comme composée d'éléments deux à deux égaux et de signes contraires. Ainsi $\iint V\frac{dx}{ds}\,d\omega=0$. Il reste donc finalement

$$\iint x\frac{dV}{ds}\,d\omega=0,$$

et l'on en conclut $x_1=0$, ce qu'il fallait démontrer.

TOUL, 4 *Juillet* 1846.

NOTE ON THE PRECEDING PAPER.

By WILLIAM THOMSON.

[*Extracted from a Letter to* M. LIOUVILLE.]

165. "...The demonstration which you have given has led me to this other theorem, that the mass M, and the shell surrounding it, have the same principal axes, through any point.

To demonstrate this, let $U=yz$ in the formula which you have given. Then, since, if we denote by K the constant value of V at the shell, we have

$$\iint V\frac{dU}{ds}\,d\omega=K\iint\frac{dU}{ds}=0,^*$$

we find

$$\iint yz\frac{-dV}{ds}\,d\omega=4\pi\iiint yz\,f(x,y,z)\,dx\,dy\,dz \qquad (1),$$

which proves the proposition enunciated.

If we take $U=x^2$, we find

$$V\frac{d^2U}{dx^2}-U\frac{d^2V}{dx^2}+V\frac{d^2U}{dy^2}-U\frac{d^2V}{dy^2}+V\frac{d^2U}{dz^2}-U\frac{d^2V}{dz^2}=2V+4\pi x^2f(x,y,z):$$

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[*] See XII. below, § 200, (8).

from which, observing that

$$\iint V\left(\frac{dU}{dx}dy\,dz+\frac{dU}{dy}dz\,dx+\frac{dU}{dz}dx\,dy\right)=K\iiint\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right)dxdydz$$
$$=2K\iiint dxdydz\,;$$

we deduce

$$\frac{1}{4\pi}\iint x^2\frac{-dV}{ds}d\omega=\frac{1}{2\pi}\iiint(V-K)dxdydz+\iiint x^2f(x,y,z)dxdydz.$$

Let A, B, C be the moments of inertia of the mass M round the axes of co-ordinates, and A_1, B_1, C_1 those of the shell, round the same axes, it being supposed that the quantity of matter of the shell is the same as that of M;[*] the preceding equation, and the two others which correspond relatively to the axes of y and z, are with this notation,

$$A_1=Q+A,\ B_1=Q+B,\ C_1=Q+C \qquad (2),\text{†}$$

where $$Q=\frac{1}{2\pi}\iiint(V-K)dxdydz,$$

is a quantity which is independent of the position of the origin.

From equations (2), we have

$$B-C=B_1-C_1,\ C-A=C_1-A_1,\ A-B=A_1-B_1 \qquad (3).$$

A demonstration of your theorem and of the theorems expressed by the equations (1) and (3) may be arrived at by comparing the expressions for the equal potentials[‡] produced by the mass M, and the shell at very distant points."[‖]

ST. PETER'S COLLEGE, *July* 15, 1846.

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[*] In this case the "density" of the distribution at any point of the shell will be equal to $\frac{1}{4\pi}\cdot\frac{-dV}{ds}$. See I. above, § 7.

[†] If the origin be taken at the centre of gravity, and the axes of co-ordinates principal axes of M (and therefore of the shell, according to the proposition enunciated above), these equations show that the "central ellipsoid" (see note to p. 202 of *Cambridge and Dublin Mathematical Journal*, 1846) for the shell is confocal with that for the body M.

[‡] A shell constructed round the mass M, in the manner described by M. Liouville, with a quantity of matter equal to M, exerts the same force upon points without the shell, as was proved first by Green (see also I. above, § 9); and since the potential of each vanishes at an infinite distance, it follows that the two bodies produce equal potentials at every point without the shell.

[‖] [See Thomson and Tait's *Natural Philosophy*, § 539.]

XI. ON CERTAIN DEFINITE INTEGRALS SUGGESTED BY PROBLEMS IN THE THEORY OF ELECTRICITY.

[From the *Cambridge and Dublin Mathematical Journal*, March 1847.]

166. It follows from the solution of the problem of the distribution of electricity on an infinite plane,* subject to the influence of an electrical point, that the value of the double integral,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z\,d\xi\,d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}\{(\xi-x')^2+(\eta-y')^2+z'^2\}^{\frac{1}{2}}},$$

is

$$\frac{2\pi}{\{(x-x')^2+(y-y')^2+(z+z')^2\}^{\frac{1}{2}}}.$$

A direct analytical verification of this result is therefore interesting in connexion with the physical problem. In the following paper the multiple integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\frac{u\,d\xi_1\,d\xi_2\ldots d\xi_s}{\{(\xi_1-x_1)^2+(\xi_2-x_2)^2+\ldots+u^2\}^{\frac{1}{2}(s+1)}\{(\xi_1-x_1')^2+(\xi_2-x_2')^2+\ldots+u'^2\}^{\frac{1}{2}(s-1)}}$$

is considered, and its value is shown to be

$$\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\{(x_1-x_1')^2+(x_2-x_2')^2+\ldots+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

a result of which the one mentioned above is a particular case. Several distinct demonstrations of this theorem are given, and some other formulæ, which have occurred to me in connexion with it, are added.

167. The first part of the following paper, which is a translation, with slight alterations, of a memoir in *Liouville's Journal*,† contains a demonstration suggested to me by a method followed by Green in proving the remarkable theorem in Art. (5) of his Essay on Electricity. In the second part some formulæ are given which, in the case of two variables, are such as would

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* See above, § 111, footnote.

† 1845, p. 137, "Démonstration d'un Théorème d'Analyse" (April 1845).

occur in the analysis of problems in heat and electricity, with reference to a body bounded in one direction by an infinite plane, if the methods indicated by Fourier were followed; and from them the value of the multiple integral mentioned above is deduced. In § III. the evaluation is effected by a direct process of reduction, suggested by geometrical considerations.*

PART I.

168. Let the value of the multiple integral, which, if we use a very convenient notation analogous to that of factorials, may be written thus,

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\,.$$

be denoted by U.

Let $u+u'=a$, it being understood that u and u' are taken as positive. Then, if we assume

$$R=\frac{1}{\{\Sigma(\xi-x)^2+v^2\}^{\frac{1}{2}(s-1)}}-\frac{1}{\{\Sigma(\xi-x)^2+(2u-v)^2\}^{\frac{1}{2}(s-1)}} \quad (1),$$

$$R'=\frac{1}{\{\Sigma(\xi-x')^2+(a-v)^2\}^{\frac{1}{2}(s-1)}} \quad (2),$$

we have

$$-2(s-1)uU=\left[\int_{-\infty}^{\infty}\right]^s R'\frac{dR}{dv}[d\xi]^s, \text{ when } v=u.$$

It is easily seen that the second member of this equation vanishes when $v=\pm\infty$, and that it does not become infinite, even when one of the values 0, $2u$, or a is assigned to v. Hence the preceding equation may be written

$$-2(n-1)uU=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]\left(\frac{dR'}{dv}\frac{dR}{dv}+R'\frac{d^2R}{dv^2}\right)[d\xi]^s dv.$$

But we have

$$\int\left[\int_{-\infty}^{\infty}\right]^s\frac{dR'}{dv}\frac{dR}{dv}[d\xi]^s dv=\left[\int_{-\infty}^{\infty}\right]^s\int\frac{dR'}{dv}\frac{dR}{dv}dv[d\xi]^s$$

$$=\left[\int_{-\infty}^{\infty}\right]^s R\frac{dR'}{dv}[d\xi]^s-\int\left[\int_{-\infty}^{\infty}\right]^s R\frac{d^2R'}{dv^2}[d\xi]^s dv.$$

When we take the integral with respect to v between the limits $-\infty$ and u, the first term vanishes, since at each limit $R=0$. Thus the preceding equation is reduced to

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* See "Extrait d'une lettre à M. Liouville, etc." Liouville's *Journal*, 1845, p. 364 (XIV. § 210, below).

$$-2(s-1)uU=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}\left(R'\frac{d^2R}{dv^2}-R\frac{d^2R'}{dv^2}\right)[d\xi]^s dv.$$

169. Now we have $\dfrac{d^2R'}{dv^2}+\Sigma\dfrac{d^2R'}{d\xi^2}=0,$

for all values of ξ_1, ξ_2..., provided v be not equal to a. Hence this equation is satisfied for all the values of the variables between the limits of the integration in the preceding expression, and we may therefore employ it to eliminate $\dfrac{d^2R'}{dv^2}$: we thus obtain

$$-2(s-1)uU=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}\left(R'\frac{d^2R}{dv^2}+R\Sigma\frac{d^2R'}{d\xi^2}\right)[d\xi]^s dv.$$

Taking one of the terms of the second member, and integrating by parts, we have

$$\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}R\frac{d^2R'}{d\xi_1{}^2}[d\xi]^s dv$$

$$=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s-1}\left(\int_{-\infty}^{\infty}R\frac{d^2R'}{d\xi_1{}^2}d\xi_1\right)[d\xi]^{s-1}dv$$

$$=-\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s-1}\left(\int_{-\infty}^{\infty}\frac{dR'}{d\xi_1}\frac{dR}{d\xi_1}d\xi_1\right)[d\xi]^{s-1}dv$$

$$=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}R'\frac{d^2R}{d\xi_1{}^2}[d\xi]^s dv,$$

since the integrated parts vanish at each limit. By applying a similar process to each term under the sign Σ, we find

$$-2(s-1)uU=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}R'\left(\frac{d^2R}{dv^2}+\Sigma\frac{d^2R}{d\xi^2}\right)[d\xi]^s dv.$$

But, if we denote by Q and Q' the two parts of R, in equation (1), so that $R=Q-Q'$, we have

$$\frac{d^2Q'}{dv^2}+\Sigma\frac{d^2Q'}{d\xi^2}=0$$

for all values of the variables v, ξ_1, etc., within the limits of integration; hence there remains

$$-2(s-1)uU=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}R'\left(\frac{d^2Q}{dv^2}+\Sigma\frac{d^2Q}{d\xi^2}\right)[d\xi]^s dv.$$

To determine the value of this expression it may be remarked that the quantity under the integral signs vanishes for all values of the variables which differ sensibly from those expressed by

$$v=0,\quad \xi_1=x_1,\quad \xi_2=x_2,\quad \text{etc.},$$

and moreover, that if we consider separately the terms of the second member, each is found to be a converging integral: it follows that, if we denote by P the value which R' receives when the variables have these values assigned, we have

$$-2(s-1)uU=P\iiint\cdots\left(\frac{d^2Q}{d^2v}+\Sigma\frac{d^2Q}{d\xi^2}\right)dv\,d\xi_1\,d\xi_2\ldots d\xi_s \qquad (3),$$

where the limits of integration must be such as to include the values 0, x_1, x_2, etc., but are otherwise entirely arbitrary. By considering separately the different terms of this expression, and integrating each with respect to the variable to which it is related, without yet assigning the limits of the integration, we find

$$-2(s-1)uU=P\left(\iint\cdots\frac{dQ}{dv}\,d\xi_1\,d\xi_2\ldots+\iint\frac{dQ}{d\xi_1}\,dv\,d\xi_2+\text{etc.}\right) \quad (4).$$

170. Let us now assume

$$\xi_1=v_1+x_1,\quad \xi_2=v_2+x_2,\quad \text{etc.},$$

and

$$v^2+v_1^2+\ldots+v_s^2=r^2,$$

from which we have

$$Q=\frac{1}{r^{s-1}},\quad \frac{dQ}{dv}=-\frac{s-1}{r^{s+1}}v,\quad \frac{dQ}{d\xi_1}=-\frac{s-1}{r^{s+1}}v_1,\quad \text{etc.}$$

The integrations in equation (3) may be extended to all the values of the variables which satisfy the condition

$$v^2+v_1^2+v_2^2+\ldots+v_s^2 \gtreqless a^2;$$

and the limits in (4) will then be such as to include all the values which satisfy the equation

$$v^2+v_1^2+v_2^2+\ldots+v_s^2=a^2,\ \text{or}\ r^2=a^2,\ \text{etc.}$$

If in the integrations we only take the positive values of the variables v, v_1, v_2, etc., which satisfy the limiting condition, we must multiply each integral by 2^{s+1}; and we may then simply take, in the successive terms the second member of (4),

$$\frac{dQ}{dv}=-\frac{s-1}{a^{s+1}}v,\quad \frac{dQ}{d\xi_1}=-\frac{s-1}{a^{s+1}}v_1,\quad \text{etc.}$$

Thus we have

$$uU=\frac{2^sP}{a^{s+1}}(\iint\ldots v\,dv_1dv_2\ldots dv_s+\iint\ldots v_1dv\,dv_2\ldots dv_s+\text{etc.})^*$$

$$=\frac{2^s(s+1)P}{a^{s+1}}\iint\ldots(a^2-v_1^{\,2}-v_2^{\,2}-\ldots-v_s^{\,2})^{\frac{1}{2}}dv_1dv_2\ldots dv_s$$

$$=(s+1)P\iint\ldots(1-l_1-l_2\ldots-l_s)^{\frac{1}{2}}l_1^{-\frac{1}{2}}l_2^{-\frac{1}{2}}\ldots l_s^{-\frac{1}{2}}dl_1dl_2\ldots dl_s\,;$$

in which last expression the limits include all positive values satisfying the condition

$$l_1+l_2+\ldots+l_s\leqq 1.$$

Hence, by Liouville's theorem,[†]

$$uU=(s+1)P\frac{\Gamma(\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\int_0^1(1-h)^{\frac{1}{2}}h^{\frac{1}{2}s-1}dh=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}P,$$

which gives the required value of the integral U.

171. If we denote by U' any integral corresponding to U, in which the system of variables u, x_1, x_2 ... and u', x_1', x_2' ... are inverted, we shall have $uU=u'U'$, since P is a function symmetrical with respect to the two systems; and we therefore deduce from the preceding result,

$$\left.\begin{aligned}&u\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\\&=u'\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}}\\&=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}}\end{aligned}\right\}\quad(5).$$

172. I shall add another demonstration of this theorem, as an application of some remarkable analysis given by Mr. Green in his memoir "On the determination of the exterior and interior attractions of ellipsoids of variable densities."[‡]

$$\text{Let }V=\left[\int_{-\infty}^{\infty}\right]^s\frac{u[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\quad(6),$$

an integral which may also be expressed thus:

$$-\frac{1}{s-1}\,\frac{d}{du}\left\{\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\right\}.$$

From this latter form, we see that the equation

$$\frac{d^2V}{du^2}+\Sigma\frac{d^2V}{dx^2}=0\qquad(7)$$

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[*] [By putting, in this, $v=\int dv$; $v_1=\int dv_1$; etc., we have

$$uU=\frac{2^sP}{a^{s+1}}(s+1)\iiint\ldots dv\,dv_1dv_2\ldots:$$

whence immediately, by a simpler case of Liouville's theorem than in the text, or by Green's transformation (see § 186), the same result.]

[†] See Gregory's *Examples* (Ed. 1841), p. 469.

[‡] Read at the Cambridge Phil. Soc., May 6, 1833. See *Trans.* of that date.

is satisfied, provided u does not vanish. Hence V is a function which satisfies this equation for all values of $x_1, x_2 \ldots$ and for all the values of u between 0 and ∞. At these limits the value of V may be easily determined, and the general value inferred in the following manner:—

173. When $u=0$, the quantity under the signs of integration in the expression for V vanishes for all the values of $\xi_1, \xi_2 \ldots$ which are not equal to $x_1, x_2 \ldots$ respectively. Hence it follows that, when $u=0$,

$$V=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\left[\int_{-\infty}^{\infty}\right]^s\frac{u[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\left[\int_{-\infty}^{\infty}\right]^s\frac{dz_1dz_2\ldots dz_s}{(1+z_1^2+z_2^2+\ldots+z_s^2)^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}},\int_0^{\infty}\int_0^{\infty}\cdots\frac{l_1^{-\frac{1}{2}}l_2^{-\frac{1}{2}}\ldots dl_1dl_2\ldots}{(1+l_1+l_2+\ldots+l_s)^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\int_0^{\infty}\frac{h^{\frac{1}{2}s-1}dh}{(1+h)^{\frac{1}{2}(s+1)}}$$

$$=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}.$$

Also, when $u=\infty$, the value of V is nothing.

174. Thus we see that V has the same value as the expression

$$\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\cdot\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

when $u=0$, and when $u=\infty$; which enables us to infer that

$$V=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\cdot\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

for all positive values of u, provided u' be taken as positive; for the second member of this equation satisfies equation (7) for all positive values of u, and for any values of the other variables, and at the limits $u=0$ and $u=\infty$ has the same value as V, and therefore, by a theorem of Green's,* in the memoir referred to, must be equal to V for all positive values of u.

175. From what has been proved above we may deduce the solution of the following problem:—

Having given for all values of $\xi_1, \xi_2 \ldots$, the value of the multiple integral

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* [Included in *Theorem* 2. of XIII. below.]

$$\mathrm{S}\frac{\rho' dx_1' dx_2' \ldots dx_s'}{\{(x_1-\xi_1)^2+(x_2'-\xi_2)^2+\ldots+(x_s'-\xi_s)^2+u'^2\}^{\frac{1}{2}(s-1)}} \qquad (a),$$

where u' and ρ' are any unknown functions of x_1', $x_2' \ldots x_s'$, let it be required to find the value of

$$\mathrm{S}\frac{\rho' dx_1' dx_2' \ldots dx_s'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+\ldots+(x_s'-x_s)^2+(u'+u)^2\}^{\frac{1}{2}(s-1)}} \qquad (b),$$

where x_1, $x_2 \ldots x_s$ are any given quantities, and u a given positive quantity.

Denoting the expression (a) by Φ, and the expression (b) by ϕ, we have, from the theorem established above,

$$\phi = \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}}\mathrm{S}\,\rho' dx_1' dx_2' \ldots dx_s'.$$

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}}\cdot\mathrm{S}\frac{\rho' dx_1' dx_2 \ldots dx_s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}},$$

or

$$\phi = \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{\Phi[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}} \qquad (c).$$

But, by hypothesis, Φ is given for all values of ξ_1, $\xi_2 \ldots \xi_s$; and therefore this equation expresses the solution of the problem. We may also deduce from the theorem (5) the expression

$$\phi = -\frac{\Gamma\frac{1}{2}(s+1)}{(s-1)\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{\Psi[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}} \qquad (d),$$

by means of which ϕ may be determined when the value, Ψ, of $\frac{d\phi}{du}$ corresponding to $u=0$ is given.

176. For the particular case of $u'=0$, the theorem (d) is included in a theorem given by Green, in which the number n in the exponent of the denominator may differ from the number s, of variables, the sole condition being that $n-s+1$ must be positive; but it is only in the case of $n=s$ that a general theorem such as (d), by means of which the general value of ϕ is obtained from the value $\frac{d\phi}{du}$ when $u=0$, can be established.

177. Let us now apply these formulæ to the case of $s=2$:

we may in this case conveniently replace x_1, x_2, z by x, y, z, and ξ_1, ξ_2, by ξ, η. Equations (c) and (d) become

$$\phi=\frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Phi d\xi d\eta}{\{(\xi-x)^2+(\eta-\gamma)^2+z^2\}^{\frac{3}{2}}} \qquad (e),$$

and

$$\phi=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Psi d\xi d\eta}{\{(\xi-x)^2+(\eta-\gamma)^2+z^2\}^{\frac{1}{2}}} \qquad (f),$$

where Ψ denotes the value of $\frac{d\phi}{dz}$ when $x=\xi$, $y=\eta$, $z=0$.

178. The first of these theorems may be deduced from a very general theorem given by Green in his essay on Electricity and Magnetism [§ (5.) eq. (6)]. The second may be demonstrated in the following manner:—

Let x', y', z' be considered as the co-ordinates of a point P', where there is situated a quantity of matter $\rho'\,dx'\,dy'\,dz'$, in the volume $dx'\,dy'\,dz'$. Then ϕ will be the potential on a point P (x, y, z), above the plane of x, y which we may regard as horizontal, due to a quantity of matter,

$$M, (=\iiint\rho'\,dx'\,dy'\,dz')$$

situated below this plane. Now it follows from a theorem, first, so far as I am aware, given by Gauss, for a surface of any form, that there is a determinate distribution of matter upon the plane (xy) which will produce this same potential on points above the plane. Let k be the density of this distribution at a point Π (ξ, η) of the plane, so that

$$\phi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{kd\xi d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{1}{2}}},$$

which gives

$$\frac{d\phi}{dz}=-z\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{kd\xi d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}}.$$

Let $z=0$; then denoting by k and $\left(\frac{d\phi}{dz}\right)_0$ the values of k and $\frac{d\phi}{dz}$ at the point (x, y, 0), we find

$$\left(\frac{d\phi}{dz}\right)_0=-\text{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{zd\xi d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}},$$

$$=-\text{k}.2\pi,$$

since the value of the integral in the second member is

2π, whatever be the value of z. Hence we conclude that $k=-\frac{1}{2\pi}.\Psi$, and equation (f) is established.

179. It should be remarked that the total quantity of matter distributed over the plane xy must be equal to the mass M, which it represents: this is readily verified from the preceding formulæ.

180. The same formulæ admit of an interesting application in the theory of heat. Thus let ϕ be the permanent temperature of a point P in an infinite homogeneous solid, heated by constant sources distributed below the plane (xy), (the case in which some of the sources are in this plane being of course included). If the temperature Φ at any point Π in the plane (xy) be given, the formula (e) enables us to find the temperature at any point above the plane.

181. As an example, let us suppose that the sources of heat are such that the temperature of a portion A of the plane (xy), between two lines parallel to OY and at equal distances, a, on its two sides, has a constant value c, and the temperature of the remainder of the plane zero. In this case the formula (e) will give, for the temperature at a point (x, y, z) above the plane,

$$\phi=\frac{zc}{2\pi}\int_{-\infty}^{\infty}\int_{-a}^{a}\frac{d\xi d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}}$$

$$=\frac{c}{\pi}\left(\tan^{-1}\frac{x+a}{z}-\tan^{-1}\frac{x-a}{z}\right)$$

$$=\frac{c}{\pi}\tan^{-1}\frac{2ax}{x^2+z^2-a^2}.$$

From this we conclude that the isothermal surfaces which correspond to this case are circular cylinders, which intersect the plane (xy) in the two parallel lines bounding A.

The application to this example, and all others in which the isothermal surfaces are cylindrical, may be made directly by putting $s=1$ in the general formulæ.

PART II.

182. I now proceed to find the values, which will be denoted by V and W, of the integrals

$$\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s[\cos m\xi]^s}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}}$$

and $$\left[\int_{-\infty}^{\infty}\right]^{s}[dm]^{s}[\cos mx]^{s}\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}},$$

where the symbols $[\cos m\xi]^s$, $[\cos mx]^s$ denote the products

$$\cos m_1\xi_1 . \cos m_2\xi_2 . . \cos m_s\xi_s,$$
$$\cos m_1x_1 . \cos m_2x_2 . . \cos m_sx_s;$$

and the notation is in other respects the same as before.

By means of the formula

$$[\cos m\xi+\sin m\xi . \sqrt{(-1)}]^s=\cos(\Sigma m\xi)+\sin(\Sigma m\xi) . \sqrt{(-1)},$$

it is easily shown that

$$V=\left[\int_{-\infty}^{\infty}\right]^{s}\frac{[d\xi]^s\cos\Sigma(m\xi)}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}} \qquad (a).$$

Hence, by a suitable linear transformation, in which one of the assumptions is $\Sigma m\xi=\eta(\Sigma m^2)^{\frac{1}{2}}$, we have [if μ denote $(\Sigma m^2)^{\frac{1}{2}}$]

$$V=\int_{-\infty}^{\infty}\cos\mu\eta : d\eta . \left[\int_{-\infty}^{\infty}\right]^{s-1}\frac{[d\zeta]^{s-1}}{(u^2+\eta^2+\Sigma\zeta^2)^{\frac{1}{2}(s-1)}} \qquad (b).$$

Now, by means of Liouville's theorem,* we find

$$\left[\int_{-\infty}^{\infty}\right]^{s-1}\frac{[d\zeta]^{s-1}}{(u^2+\eta^2+\Sigma\zeta^2)^{\frac{1}{2}(s-1)}}=\frac{\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\frac{2\xi d\xi . \xi^{s-3}}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}}.$$

Hence $$V=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\infty}\frac{\xi^{s-2}\cos\mu\eta : d\xi d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}} \qquad (c).$$

Differentiating with respect to u, by which the further reduction of the integral will be facilitated, we have

$$-\frac{dV}{du}=(s-1)\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\infty}\frac{u . \xi^{s-2}\cos\mu\eta . d\xi d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} \qquad (d).$$

Now

$$\int_0^{\infty}\frac{\xi^{s-2}d\xi}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}}=\int_0^{\infty}\frac{\xi^{-3}d\xi}{\left(1+\frac{\eta^2+u^2}{\xi^2}\right)^{\frac{1}{2}(s+1)}}=\frac{1}{2}\int_0^{\infty}\frac{dt}{\{1+(\eta^2+u^2)t\}^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{s-1} . \frac{1}{\eta^2+u^2}.$$

Hence $$-\frac{dV}{du}=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\frac{u\cos\mu\eta d\eta}{\eta^2+u^2} \qquad (e).$$

$$=\frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)} . \epsilon^{-\mu u}.$$

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* See *Cambridge Mathematical Journal*, Feb. 1841, p. 221 [or Gregory's *Examples*, Ed. 1841, p. 469].

From this, by integration with respect to u, we deduce the value of V: thus we have the result

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s[\cos m\xi]^s}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}}=\frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)}\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}} \qquad (V).$$

183. To evaluate the integral W we may in the first place reduce it to a double integral by a process similar to that indicated above, for obtaining the expression (c); and we thus find

$$W=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\infty}\frac{dm\,dn.\,m^{s-2}\cos(nr).\,\epsilon^{-(m^2+n^2)^{\frac{1}{2}}u}}{(m^2+n^2)^{\frac{1}{2}}} \qquad (a).$$

where r denotes $(\Sigma x^2)^{\frac{1}{2}}$. If we take $m=\rho\cos\vartheta$, $n=\rho\sin\vartheta$, this becomes

$$W=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} d\theta\, d\rho\, \rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\sin\vartheta)\epsilon^{-\rho u} \qquad (b).$$

Now we have

$$\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^f\cos(r\rho\sin\vartheta)\epsilon^{-\rho u}=\rho^{2f}\cos^{2f}\vartheta.\cos(r\rho\cos\vartheta)\epsilon^{-\rho u} \qquad (c).$$

Considering first the case where s is even, let $f=\frac{1}{2}s-1$; we thus find

$$\rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\cos\vartheta)\epsilon^{-\rho u}=\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\epsilon^{-\rho u},$$

and, by substitution in (b), we have

$$\begin{aligned}
W&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho.\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\epsilon^{-\rho u},\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\cos(r\rho\sin\vartheta)\epsilon^{-\rho u}\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^{\frac{1}{2}\pi}\frac{u\,d\vartheta}{u^2+r^2\sin^2\vartheta}\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\frac{\frac{1}{2}\pi}{(u^2+r^2)^{\frac{1}{2}}}\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s+1)}\frac{1^2.3^2\ldots(s-1)^2}{(u^2+r^2)^{\frac{1}{2}(s-1)}}=2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}} \qquad (a).
\end{aligned}$$

In the second case, when s is odd, let $f=\frac{1}{2}(s-1)$ in (c); then, making use of the result in (b), we have

$$\begin{aligned}
W&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\,\rho\cos\vartheta.\cos(r\rho\sin\vartheta)\epsilon^{-\rho u}\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\int_0^{\infty} d\rho\cdot\frac{\sin(r\rho)}{r}\epsilon^{-\rho u}\\
&=\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\frac{1}{r^2+\rho^2}=2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}},
\end{aligned}$$

Hence, whether s be odd or even, we conclude that

$$\left[\int_0^\infty\right]^s [dm]^s[\cos mx]^s \frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}} = 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+\Sigma x^2)^{\frac{1}{2}(s-1)}} \quad (W).$$

184. The investigation which we have just gone through, of the integrals (V), (W) constitutes the verification of "Fourier's theorem" in a particular case. For, by this theorem, we have, if $F(x_1, x_2 \ldots)$ be a function which remains the same when the signs of any of the variables are changed,

$$2^s\pi^s F(x_1, x_2 \ldots) =$$
$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s[\cos mx]^s \int_{-\infty}^{\infty}[d\xi]^s[\cos m\xi]^s F(\xi_1, \xi_2 \ldots) \quad (\epsilon):$$

and if we take

$$F(\xi_1, \xi_2 \ldots) = \frac{1}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}},$$

the result of the integrations with respect to ξ_1, $\xi_2 \ldots$, is given by (V), and the second member thus becomes a multiple integral with respect to m_1, $m_2 \ldots$, which is shown by (W) to be equal to the first member. Conversely, if we assume Fourier's theorem, we may deduce the value W, by means of it, from that of V. The integrals V and W are also connected by means of another case of Fourier's theorem, found by taking, in (ϵ),

$$F(\xi_1, \xi_2 \ldots) = \frac{\epsilon^{-(\Sigma\xi^2)^{\frac{1}{2}}u}}{(\Sigma\xi^2)^{\frac{1}{2}}}.$$

In this way, after the value of W has been found, that of V may be deduced.

185. The formulæ (V) and (W) may be applied to evaluate the multiple integral u, and we shall thus obtain the result of the investigation in § I. in a different manner.

By means of the equation obtained by differentiating (W) with respect to u, we find

$$\frac{u}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}} = \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)}$$
$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s[\cos m(\xi-x)]^s \epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u};$$

Making this substitution, for one of the factors of the expression under the integral signs U, we have

$$Uu=\frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)}\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}$$

$$\vdots\left[\int_{-\infty}^{\infty}\right][dm]^s[\cos m(\xi-x)]^s\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}$$

$$=\frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s[dm]^s[\cos m(x-x')]^s\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s[\cos m(\xi-x')]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}$$

$$=\frac{\pi}{2^{s-2}(s-1)\{\Gamma\frac{1}{2}(s-1)\}^2}$$

$$\left[\int_{-\infty}^{\infty}\right]^s[dm]^s[\cos m(x-x')]^s\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u'}}{(\Sigma m^2)^{\frac{1}{2}}},\text{ by }(V),$$

$$=\frac{2\pi^{\frac{1}{2}(s+1)}}{(s-1)\Gamma\frac{1}{2}(s-1)}\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},\text{ by }(W),$$

which agrees with the value obtained above.

PART III.

186. The value of the integral U may also be obtained by a direct process of reduction, as follows :—

By a suitable linear transformation, in which assumptions such as

$$\xi_1-x_1=\Sigma a\zeta$$

are made, we find

$$U=\left[\int_{-\infty}^{\infty}\right]\frac{[d\zeta]^s}{(\Sigma\zeta^2+u^2)^{\frac{1}{2}(s+1)}(\Sigma\zeta^2-2f\zeta_1+f^2+u'^2)^{\frac{1}{2}(s-1)}}\qquad(a).$$

where

$$f^2=\Sigma(x-x')^2.$$

Let us now assume

$$\zeta_1=\rho\cos\phi,\ \zeta_2=\rho\sin\phi\cos\theta_1,\ \zeta_3=\rho\sin\phi\sin\theta_1\cos\theta_2\ldots,$$
$$\zeta_{s-1}=\rho\sin\phi\sin\theta_1\sin\theta_2\ldots\ldots\cos\theta_{s-2},$$
$$\zeta_s=\rho\sin\phi\sin\theta_1\sin\theta_2\ldots\ldots\sin\theta_{s-2},$$

from which we deduce*

$$[d\xi]^s=\rho^{s-1}\sin^{s-2}\phi\sin^{s-3}\theta_1\sin^{s-4}\theta_2\ldots\ldots\sin\theta_{s-3}[d\theta]^{s-2}d\phi d\rho\,;$$

a transformation given first by Green. Equation (a) is thus reduced to

$$U=H_{s-2}\int_0^{\infty}\int_0^{\pi}\frac{\rho^{s-1}\sin^{s-2}\phi d\phi d\rho}{(\rho^2+u^2)^{\frac{1}{2}(s+1)}(\rho^2-2\rho f\cos\phi+f^2+u'^2)^{\frac{1}{2}(s-1)}}\ldots\ (b).$$

where H_{s-2} denotes the product

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* See *Cambridge Mathematical Journal*, Nov. 1843, p. 24, First Series; [or Green, "Attraction of Ellipsoids," § 6, *Camb. Phil. Trans.*, May 1833.]

$$\int_0^\pi \sin^{s-2}\theta d\theta . \int_0^\pi \sin^{s-3}\theta d\theta \ldots\ldots \int_0^{2\pi} d\theta .$$

Let $\rho = u \tan \frac{1}{2}\vartheta$; we thus get

$$Uu = \tfrac{1}{2} H_{s-2} \int_0^\pi \int_0^\pi$$

$$\frac{\sin^{s-1}\vartheta \sin^{s-2}\phi d\phi d\vartheta}{\{2(f^2+u'^2+u^2)+2(f^2+u'^2-u^2)\cos\vartheta-4uf\sin\vartheta\cos\phi\}^{\frac{1}{2}(s-1)}}$$

and we may now conveniently assume

$$2(f^2+u'^2+u^2)=h^2+k^2,$$

$$2(f^2+u'^2-u^2)\cos\vartheta-4uf\sin\vartheta\cos\phi$$

$$=2\{(f^2+u^2-u'^2)^2+4u^2f^2\}^{\frac{1}{2}}\cos\theta=2hk\cos\theta,$$

and $\sin\phi\sin\vartheta=\sin\varphi\sin\theta,$

from which we deduce

$$h^2=(u'+u)^2+f^2,\ k^2=(u'-u)^2+f^2,$$

$$\sin\vartheta d\phi d\vartheta=\sin\theta d\varphi d\theta;$$

the expression for U becomes

$$Uu=\tfrac{1}{2}H_{s-2}\int_0^\pi\int_0^\pi \frac{\sin^{s-1}\theta\sin^{s-2}\varphi d\varphi d\theta}{(h^2-2hk\cos\theta-k^2)^{\frac{1}{2}(s-1)}}$$

$$=\tfrac{1}{2}H_{s-1}\int_0^\pi \frac{\sin^{s-1}\theta d\theta}{(h^2-2hk\cos\theta+k^2)^{\frac{1}{2}(s-1)}}$$

Let $h\sin(\psi-\theta)=k\sin\psi;$

by means of this transformation, observing that $h>k$, we readily find

$$Uu=\frac{\frac{1}{2}H_s}{h^{s-1}}$$

or $$Uu=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

which is the same as the result previously obtained.

ST. PETER'S COLLEGE, *Oct.* 2, 1846.

XII. PROPOSITIONS IN THE THEORY OF ATTRACTION.

[From the *Camb. Math. Jour.*, Nov. 1842 and Feb. 1843.]

187. Let x, y, z be the co-ordinates of any point P in an attracting or repelling body M; let dm be an element of the mass, at the point P, which will be positive or negative according as it is attractive or repulsive; let x', y', z' be the co-ordinates of an attracted point P'; let

$$\Delta = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}};$$

and let

$$v' = \int \frac{dm}{\Delta},$$

the integral including the whole of M. This expression has been called by Green the potential* of the body M, on the point P', and the same name has been employed by Gauss (in a Mémoire on "General Theorems relating to Attractive and Repulsive Forces, in the *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1839, Leipsic 1840, edited by M. Gauss and Weber).† By a known theorem, the components of the attraction of M on P', in the directions of x, y, z, are

$$-\frac{dv'}{dx'}, \quad -\frac{dv'}{dy'}, \quad -\frac{dv'}{dz'},$$

and if $d\gamma'$ be the element of any line, straight or curved, which passes through P', the attraction in the direction of this element is $\frac{dv'}{d\gamma'}$. Hence it follows that if a surface be drawn through any point P' for every point of which the potential has the same value, the attraction on every point in the surface is wholly in the direction of the normal. Surfaces for which the potential is constant are therefore called, by Gauss, *surfaces of equilibrium*. It has been shown in a former paper (I. above), that if M, instead of an attractive mass, were a group of sources of heat or cold in the interior of an infinite homogeneous solid, v' would be the permanent temperature produced by them at P'. In that case, the surfaces of equilibrium would be *isothermal surfaces.*

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* ["This I found in a reference to his memoirs, in Murphy's first memoir "on definite integrals. Ever since I have been trying to see Green's memoir, "but could not hear of it from anybody till to-day, when I have got a copy "from Mr. Hopkins. Jan. 25, 1845." (Private note which I find written on p. 190 of vol. iii. of my copy of the *Camb. Math. Jour.*)]

† Translations of this paper have been published in Taylor's *Scientific Memoirs* for April 1842, and in the Numbers of Liouville's Journal for July and August 1842.

188. When the attraction of (positive or negative) matter, as for instance electricity, spread over a surface is considered, the density of the matter at any point is measured by the quantity of matter on an element of the surface, divided by that element.

The principal object of this paper is to prove the following theorems:—

If upon E, one of the surfaces of equilibrium enclosing an attracting mass, its matter be distributed in such a manner that its density at any point P is equal to the attraction of M on P; then—

(1.) The attraction of the matter spread over E, on an external point, is equal to the attraction of M on the same point multiplied by 4π.

(2.) The attraction of the matter on E, on an internal point, is nothing.

189. These theorems were proved in a previous paper (I. §§ 5, 9), from considerations relative to the uniform motion of heat; but in the following they are proved by direct integration:—

Let u be the potential of M, on the point P, (xyz) in E. The components of the attraction of M on P, in the directions of x, y, z, are

$$-\frac{du}{dx}, \quad -\frac{du}{dy}, \quad -\frac{du}{dz};$$

and hence, if α, β, γ be the angles which a normal to E at P makes with these directions, the total attraction on P is

$$-\left(\frac{du}{dx}\cos\alpha+\frac{du}{dy}\cos\beta+\frac{du}{dz}\cos\gamma\right) \text{ or } -\frac{du}{dn},$$

if dn be an element of the normal through P.

This is therefore the expression for the density at P of the matter we have supposed to be spread over E. Let ds be an element of E at P; let v' be the potential of E, on a point P', $(x'y'z')$, either within or without E; and let Δ be the distance from P to P'. Then

$$v' = -\left\{\int \frac{\left(\frac{du}{dx}\cos\alpha + \frac{du}{dy}\cos\beta + \frac{du}{dz}\cos\gamma\right)ds}{\Delta}\right\} = -\left\{\int \frac{\frac{du}{dn}ds}{\Delta}\right\} \quad (a),$$

the brackets enclosing the integrals denoting that the integrations are to be extended over the whole surface E. Now for ds, we may choose any one of the expressions,

$$ds = \frac{dydz}{\cos\alpha},\quad ds = \frac{dxdz}{\cos\beta},\quad ds = \frac{dxdy}{\cos\gamma}.$$

Hence any integral of the form

$$\{\int(A\cos\alpha + B\cos\beta + C\cos\gamma)ds\}$$

may be transformed into the sum of the three integrals,

$$(\iint Adydz),\quad (\iint Bdxdz),\quad (\iint Cdxdy),$$

by using the first, second, and third of the expressions for ds in the first, second, and third terms of the integral respectively.

Hence, if $A = \frac{d\phi}{dx}\psi,\quad B = \frac{d\phi}{dy}\psi,\quad C = \frac{d\phi}{dz}\psi,$

$$\left(\int\frac{d\phi}{dn}\psi ds\right) \text{ or } \left\{\int\left(\frac{d\phi}{dx}\cos\alpha + \frac{d\phi}{dy}\cos\beta + \frac{d\phi}{dz}\cos\gamma\right)\psi ds\right\},$$

$$= \left\{\iint\psi\left(\frac{d\phi}{dx}dydz + \frac{d\phi}{dy}dxdz + \frac{d\phi}{dz}dxdy\right)\right\} \quad (b),$$

the limits of the integrations relative to y and z, x and z, x and y, being so chosen as to include the whole of the surface considered.

190. Making use of this transformation in (a) we have

$$v' = -\left\{\iint\left(\frac{du}{dx}\frac{dydz}{\Delta} + \frac{du}{dy}\frac{dxdz}{\Delta} + \frac{du}{dz}\frac{dxdy}{\Delta}\right)\right\} \quad (a').$$

Now $$\iint\frac{du}{dx}\frac{dydz}{\Delta} = \iint dydz\int dx\left(\frac{d^2u}{dx^2}\frac{1}{\Delta} + \frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\right)$$

$$= \iiint dxdydz\left(\frac{d^2u}{dx^2}\frac{1}{\Delta} + \frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\right).$$

191. Hence, if the integrals in the second member include every point in the space contained between E, and another surface of equilibrium, $E_{,}$, without E, and which we shall suppose to be also without P', we have

$$\left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\}_{,} - \left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\} = \iiint\left(\frac{d^2u}{dx^2}\frac{1}{\Delta} + \frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\right)dxdydz,$$

the accent denoting that, in the term accented, the integrals are to be extended over the surface $E_,$. Modifying in a similar manner the second and third terms of v', we have

$$\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}_, - \left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\} \text{ or } \left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}_, + v'$$

$$= \iiint\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{du}{dx}\frac{d}{dx}\cdot\frac{1}{\Delta}+\frac{du}{dy}\frac{d}{dy}\cdot\frac{1}{\Delta}+\frac{du}{dz}\frac{d}{dz}\cdot\frac{1}{\Delta}\right)dxdydz \quad (c).$$

Now, for all points without M,

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

by a known theorem; and such points only are included in the integrals in the second member of (c).

Also, by integration by parts,

$$\iiint\frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}dxdydz = \iint u\frac{d}{dx}\frac{1}{\Delta}dydz - \iiint u\frac{d^2}{dx^2}\frac{1}{\Delta}dxdydz$$

$$= \left\{\iint u\frac{d}{dx}\frac{1}{\Delta}dydz\right\}_, - \left\{\iint u\frac{d}{dx}\frac{1}{\Delta}dydz\right\} - \iiint u\frac{d^2}{dx^2}\frac{1}{\Delta}dxdydz.$$

Modifying similarly the two remaining terms of the second member of (c), we have

$$\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}_, + v' = \left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}dydz+\frac{d}{dy}\frac{1}{\Delta}dxdz+\frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}_,$$

$$- \left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}dydz+\frac{d}{dy}\frac{1}{\Delta}dxdz+\frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}$$

$$- \iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta}+\frac{d^2}{dy^2}\frac{1}{\Delta}+\frac{d^2}{dz^2}\frac{1}{\Delta}\right)dxdydz \quad (c').$$

Now, since E and $E_,$ are surfaces of equilibrium, u is constant for each. Again,

$$\frac{d^2}{dx^2}\frac{1}{\Delta}+\frac{d^2}{dy^2}\frac{1}{\Delta}+\frac{d^2}{dz^2}\frac{1}{\Delta}=0,$$

except when P coincides with P', at which point u has the value u'. Hence, the value of the integrals,

$$\iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta}+\frac{d^2}{dy^2}\frac{1}{\Delta}+\frac{d^2}{dz^2}\frac{1}{\Delta}\right)dxdydz$$

is only affected by these elements, for which $u=u'$, and hence u may be taken without the integral sign, as being constant and equal to u'. If, therefore, for brevity, we put

$$\iint\left(\frac{d}{dx}\frac{1}{\Delta}dydz+\frac{d}{dy}\frac{1}{\Delta}dxdz+\frac{d}{dz}\frac{1}{\Delta}dxdy\right)=(h)\text{ or }(h)_, \qquad (d),$$

according as the integrals refer to E, or to $E_,$, and

$$\iiint\left(\frac{d^2}{dx^2}\frac{1}{\Delta}+\frac{d^2}{dy^2}\frac{1}{\Delta}+\frac{d^2}{dz^2}\frac{1}{\Delta}\right)dxdydz=k \qquad (e),$$

the integrations including every point between E and E'; equation (c) becomes

$$\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}_,+v'=(u)_,(h)_,-(u)(h)-u'k \qquad (c'').$$

Now it is obvious that, at a great distance from M, the surfaces of equilibrium are very nearly spherical. Let E' be taken so far off that it may be considered as spherical, without sensible error, and let γ be the distance of any point in E' from the centre, a fixed point in M, or, which is the same, the radius of the sphere. Then $-\frac{du}{dn}$, or $-\frac{du}{d\gamma}$, is the attraction of M, on a point in E', and is therefore equal to $\frac{M}{\gamma^2}$, and therefore, by the known expression for the potential of a uniform spherical shell, on an interior point,

$$-\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}_, \text{ or } \frac{M}{\gamma^2}\left\{\int\frac{ds}{\Delta}\right\}_,=\frac{M}{\gamma^2}4\pi\gamma=4\pi(u), \qquad (f).$$

It now only remains to determine the integrals (h), $(h)_,$ and k.

By putting, in (b), $\psi=1$, $\phi=\frac{1}{\Delta}$, we find the following transformation, for (h),

$$h=\int\frac{d\frac{1}{\Delta}}{dn}ds=-\int\frac{d\Delta}{dn}\frac{ds}{\Delta^2}.$$

Now let the point (xyz) be referred to the polar co-ordinates, γ, θ, ϕ. Then, if P' be pole, $\gamma=\Delta$. Also, if ψ be the angle angle between Δ and dn, the expression for ds is

$$ds=\frac{\Delta^2\sin\theta d\theta d\phi}{\cos\psi}, \text{ or, since } \cos\psi=\frac{d\Delta}{dn},$$

$$ds=\frac{\Delta^2\sin\theta d\theta d\phi}{\frac{d\Delta}{dn}}.$$

Hence $$h=-\iint\sin\theta d\theta d\phi.$$

If P' be within the surface to which the integrals refer, the limits for θ are 0 and π, and for ϕ, 0 and 2π, and in that case, $h = -4\pi$; therefore, since P' is always within $E_{,}$,

$$(h)_{,} = -4\pi \qquad (g).$$

If P' be without the surface considered, then, for each value of θ, we must take the sum of the expressions

$$-\sin\theta d\theta d\phi, \text{ and } -\sin\theta(-d\theta)d\phi,$$

and, therefore, each element of the integral is destroyed by another equal to it, but with a contrary sign, and the value of the complete integral is therefore zero.

Hence, according as P' is without or within E,

$$(h) = 0, \text{ or } (h) = -4\pi \qquad (h).$$

Again, to find the value of k, we have, by dividing it into three terms, and integrating each once,

$$k = \left\{\iint\left(\frac{d}{dx}\frac{1}{\Delta}dydz + \frac{d}{dy}\frac{1}{\Delta}dxdz + \frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}_{,}$$

$$-\left\{\iint\left(\frac{d}{dx}\frac{1}{\Delta}dydz + \frac{d}{dy}\frac{1}{\Delta}dxdz + \frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}$$

$$= (h)_{,} - (h) = -4\pi - 0, \text{ or } = -4\pi + 4\pi;$$

and, therefore, according as P' is without or within E,

$$k = -4\pi, \text{ or } k = 0 \qquad (k).$$

Hence, making use of (f), (g), (h), (k), in (c''), we have

$$v' = 4\pi u', \text{ when } P' \text{ is without } E. \qquad (1),$$

$$v' = 4\pi(u), \text{ when } P' \text{ is within } E \qquad (2).$$

From the first of these equations it follows that the attraction of E, on a point without it, is the same as that of M, multiplied by 4π; and since the second shows that the potential of E on internal points is constant, we infer that the attraction of E on internal points is nothing.

These theorems, along with some others which were also proved in the previous paper in this Journal, already referred to, had, I have since found, been given previously by Gauss. One of the most important of these is the following:—If a mass M be wholly within or wholly without a surface, an equal mass may be distributed over this surface [in the former case, or a certain less mass may be distributed over it in the latter case] in such a manner that its attraction, in the former case on external points, and in the latter on internal, will be equal to the attraction of M on the same points. This theorem, which was proved from physical considerations in the paper *On the Uniform Motion of Heat, etc.*, is proved analytically in Gauss's *Mémoire*, but the same method is used in both to infer from it the truth of propositions (1) and (2).

From Prop. (2) it follows that, if E be the surface of an electrified conducting body, the intensity of the electricity at any point will be proportional to the attraction of M on the point. Hence we have the means of finding an infinite number of forms for conducting bodies, on which the distribution of electricity can be determined.

Thus, if M consists of a group of material points, m_1, m_2, etc., whose co-ordinates are x_1, y_1, z_1; x_2, y_2, z_2, etc.: the general equation to the surfaces of equilibrium is

$$\frac{m_1}{\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\}^{\frac{1}{2}}}+\frac{m_2}{\{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2\}^{\frac{1}{2}}}+\text{etc.}=\lambda,$$

and the intensity of electricity at any point of a solid body, bounded by one of them, will be the value of

$$\left\{\left(\frac{d\lambda}{dx}\right)^2+\left(\frac{d\lambda}{dy}\right)^2+\left(\frac{d\lambda}{dz}\right)^2\right\}^{\frac{1}{2}},$$

at the point.

To take a simple case:—Let there be only two material points, of equal intensity. The surface will then be a surface of revolution, and will be symmetrical with regard to a plane perpendicular, through its point of bisection, to the line joining the two points, and would probably very easily be constructed in practice. We should thus have a simple method of verifying numerically the mathematical theory of electricity.

PART II.

[From the *Cambridge Mathematical Journal*, February 1843.]

199. I shall now prove a general theorem, which comprehends the propositions demonstrated in Part I., along with several others of importance in the theories of electricity and heat.

Let M and M_1 be two bodies, or groups or attracting or repelling points; and let v and v_1 be their potentials on xyz; let R and R_1 be their total attractions on the same point; and let θ be the angle between the directions of R and R_1, and

$\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, the angles which they make with xyz. Let S be a closed surface, ds an element, corresponding to the co-ordinates xyz; and P and P_1 the components of R, R_1, in a direction perpendicular to the surface at ds. Then we have

$$R\cos\alpha=-\frac{dv}{dx},\quad R\cos\beta=-\frac{dv}{dy},\quad R\cos\gamma=-\frac{dv}{dz},$$

$$R_1\cos\alpha_1=-\frac{dv_1}{dx},\quad R_1\cos\beta_1=-\frac{dv_1}{dy},\quad R_1\cos\gamma_1=-\frac{dv_1}{dz},$$

$$\cos\theta=\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1+\cos\gamma\cos\gamma_1;$$

hence, $$\frac{dv}{dx}\frac{dv_1}{dx}+\frac{dv}{dy}\frac{dv_1}{dy}+\frac{dv}{dz}\frac{dv_1}{dz}=RR_1\cos\theta.$$

Hence,

$$\iiint RR_1\cos\theta\,dxdydz=\iiint\left(\frac{dv}{dx}\frac{dv_1}{dx}+\frac{dv}{dy}\frac{dv_1}{dy}+\frac{dv}{dz}\frac{dv_1}{dz}\right)dxdydz \qquad (a),$$

where we shall suppose the integrals to include every point in the interior of S. Now, by integration by parts, the second member may be put under the form,

$$\iint v_1\left(\frac{dv}{dx}dydz+\frac{dv}{dy}dxdz+\frac{dv}{dz}dxdy\right)$$
$$-\iiint v_1\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)dxdydz \qquad (b);$$

where the double integrals are extended over the surface S, and the triple integrals, as before, over every point in its interior. If we transform the first term of this by (*b*), Part I., and observe that $-\frac{dv}{dn}=P$, it becomes

$$-\iint v_1Pds.$$

Again, $$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0 \qquad (c),$$

except when xyz is a point of the attracting mass.

If this be the case, and if k be the density of the matter at the point, we have

$$\left.\begin{aligned}\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+4\pi k=0\\ \text{therefore }\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)dxdydz+4\pi dm=0\end{aligned}\right\} \qquad (d).$$

Hence (*a*) is transformed into

$$\iiint RR_1\cos\theta\,dxdydz=4\pi\iiint v_1dm-\iint v_1Pds \qquad (8);$$

similarly, by performing the integration in (a), on the terms

$$\frac{dv}{dx}, \frac{dv}{dy}, \frac{dv}{dz}, \text{ instead of } \frac{dv_1}{dx}, \frac{dv_1}{dy}, \frac{dv_1}{dz},$$

we should have found

$$\iiint RR_1 \cos\theta\, dxdydz = 4\pi \iiint v\, dm_1 - \iint vP_1 ds \qquad (4).$$

200. If the triple integrals in (a) were extended over all the space without S, or over every point between S, and another surface, $S_{\prime\prime}$, enclosing it, at an infinite distance, it may be shown, as in Part I., that the superior values of the double integrals in (b), corresponding to $S_{\prime\prime}$, vanish. Hence, the inferior values being those which correspond to S, we have, instead of (3) and (4),

$$\iiint RR_1 \cos\theta\, dxdydz = 4\pi \iiint v_1 dm + \iint v_1 P ds \qquad (5)$$

$$\iiint RR_1 \cos\theta\, dxdydz = 4\pi \iiint v\, dm_1 + \iint vP_1 ds \qquad (6).$$

It is obvious that v and v_1 in these equations may be any functions, each of which satisfy equations (c) and (d), whether we consider them as potentials or temperatures, or as mere analytical functions with the restriction that, in (5) and (6), v and v_1 must be such as to make $\iint v_1 P ds$ and $\iint vP_1 ds$ vanish at S, [and (a condition the necessity for which has been discovered by Helmholtz*), that, in (3) and (4), if S be multiply continuous, v and v_1 must be single-valued functions throughout it]. If each of them satisfy (c) for all the points within the limits of the triple integrals considered, dm and dm_1 will each vanish; but if there be any points within the limits, for which either v or v_1 does not satisfy (c), the value of dm or dm_1 at those points will be found from (d).

201. Thus let $v_1 = 1$, for every point. Then we must have $dm_1 = 0$. Also $R_1 = 0$, $P_1 = 0$.

Hence (3) becomes

$$\iint P ds = 4\pi \iiint dm = 4\pi m \qquad (7),$$

if m be the part of M within S. This expression is independent of the quantity of matter without S, and if $m = 0$ it becomes

$$\iint P ds = 0 \qquad (8).$$

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* [See Helmholtz; Crelle's *Journal*, 1858 (Wirbelbewegung), translated by Tait, *Phil. Mag.* 1867, I. (Vortex-Motion); or Thomson (Vortex-Motion, §§ 54...58), *Trans. Royal Society of Edinburgh*, 1868.]

If M be a group of sources of heat in a solid body, P will be the flux across a unit of surface, at the point xyz. Hence the total flux of heat across S is equal to the sum of the expenditures from all the sources in the interior; and if there be no sources in the interior, the whole flux is nothing. Both these results, though our physical ideas of heat would readily lead us to anticipate them, are by no means axiomatic when considered analytically. In exactly a similar manner, Poisson[*] proves that the total flux of heat out of a body during an instant of time is equal to the sum of the diminutions of heat of each particle of the body, during the same time. This follows at once from (7). For if we suppose there to be no sources of heat within S, but the temperature of interior points to vary with the time, on account of a non-uniform initial distribution of heat, we have

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=\frac{dv}{dt}.$$

Hence, by (d), we must use $-\frac{dv}{dt}dxdydz$, instead of $4\pi dm$, and therefore (7) becomes

$$\iint Pds=\iiint\frac{-dv}{dt}dxdydz.$$

It was the analysis used by Poisson, in the demonstration of this theorem, that suggested the demonstrations given in Part I., of propositions (1) and (2).

202. As another example of the application of the theorem expressed by (3) and (4), let v_1 be the potential of a unit of mass, concentrated at a fixed point, $x'y'z'$. Hence, $M_1=1$ and $dm_1=0$, except when xyz, at which dm_1 is supposed to be situated, coincides with $x'y'z'$; and, if Δ be the distance of xyz from $x'y'z'$, $v_1=\frac{1}{\Delta}$.

Hence, according as $x'y'z$ is without or within S,

$$\iiint vdm_1=0, \text{ or } \iint vdm_1=v'\iiint dm_1=v' \qquad (e),$$

the triple integrals being extended over the space within S. Now let us suppose M to be such, that v has a constant value (v) at S. Then $\iint vP_1ds=(v)\iint P_1ds$, which, by (7), is $=0$, or to $4\pi(v)$, according as $x'y'z'$ is without or within S. Hence, by comparing (3) and (4), we have, in the two cases,

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[*] See *Théorie de la Chaleur*, p. 177.

$$4\pi\iiint\frac{dm}{\Delta}-\iint\frac{Pds}{\Delta}=0, \text{ or } \iint\frac{Pds}{\Delta}=4\pi\iiint\frac{dm}{\Delta}=4\pi v' \qquad (9),$$

and $$4\pi\iiint\frac{dm}{\Delta}-\iint\frac{Pds}{\Delta}=-4\pi(v)+4\pi v';$$

therefore $$\iint\frac{Pds}{\Delta}=4\pi(v) \qquad (10).$$

These are the two propositions (1) and (2) proved in Part I., which are therefore, as we see, particular cases of the general theorem expressed by (3) and (4).*

203. If $v=v_1$, and if both arise from sources situated without S, (3) becomes

$$\iiint R^2 dxdydz=\iint vPds \qquad (11),$$

a proposition given by Gauss. If v have a constant value (v) over S, we have

$$\iint vPds=(v)\iint Pds=0, \text{ by (8)},$$

hence $$\iiint R^2 dxdydz=0.$$

Therefore $R=0$ and $v=(v)$ for interior points. Hence, if the potential produced by any number of sources have the same value over every point of a surface which contains none of them, it will have the same value for every interior point also. If we consider the sources to be spread over S, it follows that $v=(v)$ at the surface is a condition which implies that the attraction on an interior point will be nothing. Hence the sole condition for the distribution of electricity over a conducting surface, is that its attraction shall be everywhere perpendicular to the surface, a proposition which was proved from indirect considerations, relative to heat, in a former paper.†

204. In exactly a similar manner, if none of the sources be without S, by means of (5) and (7), it may be shown that

$$\iiint R^2 dxdydz=4\pi M(v) \qquad (12);$$

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* It may be here proper to state that these theorems, which were first demonstrated by Gauss, are the subject of a Mémoire by M. Chasles, in the Additions to the *Connaissance des Temps* for 1845, published in June 1842. In this Mémoire he refers to an announcement of them, without a demonstration, in the *Comptes Rendus des Séances de l'Académie des Sciences*, Feb. 11, 1839, a date earlier than that of M. Gauss's Mémoire, which was read at the Royal Society of Göttingen in March 1840.

† See I. above, § 5.

the triple integrals being extended over all the space without S. Hence a quantity of matter μ can only be distributed in one way on S, so as to make (v) be constant. For if there were two distributions of μ, each making (v) constant, there would be a third, corresponding to their difference, which would also make (v) constant. The whole mass in the third case would be nothing. Hence, by (12), we must have $\iiint R^2 dxdydz = 0$, and therefore $R = 0$ for external points; and, since (v) is constant at the surface, R must be $= 0$ for interior points also. Now this cannot be the case unless the density at each point of the surface be nothing, on account of the theorem of Laplace, that, if ρ be the density at any point of a stratum which exerts no attraction on interior points, its attraction on an interior point close to the surface will be $4\pi\rho$. This important theorem, which shows that there is only one distribution of electricity on a body that satisfies the condition of equilibrium, was first given by Gauss. It may be readily extended, as has been done by Liouville,[*] to the case of any number of electrified bodies, influencing one another, by supposing S to consist of a number of isolated portions, which will obviously not affect the truth of (5) and (6).

Then, if we suppose v to have the constant values, (v), $(v)'$, etc., at the different surfaces, and the quantities of matter on these surfaces to be M, M', etc., we should have, instead of (11),

$$\iiint R^2 dxdydz = 4\pi\{M(v) + M'(v)' + \text{etc.}\} \qquad (13),$$

and from this it may be shown, as above, that there is only one distribution of the same quantities of matter, M, M', etc., which satisfies the conditions of equilibrium.

205. If both M and M_1 be wholly within S, by comparing (5) and (6), or if both be without S, by comparing (3) and (4), we have

$$\iint Pv_1 ds = \iint P_1 v ds \qquad (14).$$

Now let S be a sphere, and let $r\theta\phi$ be the polar co-ordinates, from the centre as pole, of any point in the surface to which the potentials v and v_1 correspond. Then we shall have

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[*] See Note to M. Chasles' Mémoire in the *Connaissance des Temps* for 1845.

$P=-\frac{dv}{dr}$, $P_1=-\frac{dv_1}{dr}$, and we may assume $ds=r^2\sin\theta d\theta d\phi$. Hence (14) becomes

$$\int_0^\pi\int_0^{2\pi} v_1\frac{dv}{dr}\sin\theta d\theta d\phi=\int_0^\pi\int_0^{2\pi} v\frac{dv_1}{dr}\sin\theta d\theta d\phi \qquad (15).$$

This equation leads at once to the fundamental property of Laplace's coefficients. For if v and v_1 be of the forms $Y_m r^m$, $Y_n r^n$, m and n being any positive or negative integers, zero included, and Y_m and Y_n being independent of r, we have, by substitution in (15),

$$m\int_0^\pi\int_0^{2\pi} Y_m Y_n \sin\theta d\theta d\phi=n\int_0^\pi\int_0^{2\pi} Y_m Y_n \sin\theta d\theta d\phi.$$

If m be not $=n$, this cannot be satisfied unless

$$\int_0^\pi\int_0^{2\pi} Y_m Y_n \sin\theta d\theta d\phi=0 \qquad (16).$$

This is the* fundamental property of Laplace's coefficients.

There are some other applications of the general theorem which has been established, especially to the Theory of Electricity, which must, however, be left for a future opportunity.

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* [For a justification of this use of the definite article, see Murphy's *Electricity*, Chap. I. Props. I. and II., Cambridge 1833.]

XIII. THEOREMS WITH REFERENCE TO THE SOLUTION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, Jan. 1848.]

206. *Theorem* 1. It is possible to find a function V, of x, y, z,* which shall satisfy, for all real values of these variables, the differential equation

$$\frac{d\left(a^2\frac{dV}{dx}\right)}{dx}+\frac{d\left(a^2\frac{dV}{dy}\right)}{dy}+\frac{d\left(a^2\frac{dV}{dz}\right)}{dz}=-4\pi\rho \qquad \text{(A)},$$

a being any real continuous or discontinuous function of x, y, z, and ρ a function which vanishes for all values of x, y, z, exceeding certain finite limits (such as may be represented geometrically by a finite closed surface), within which its value is finite, but entirely arbitrary.

Theorem 2. There cannot be two different solutions of equation (A) for all real values of the variables.

1. (*Demonstration*).—Let U be a function of x, y, z, given by the equation

$$U=\iiint\frac{\rho'dx'dy'dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}} \qquad (a),$$

the integrations in the second member including all the space for which ρ' is finite; so that, if we please, we may conceive the limits of each integration to be $-\infty$ and $+\infty$, as thus all the values of the variables for which ρ' is finite will be included, and the amount of the integral will not be affected by those values of the variables for which ρ' vanishes, being included. Again, V being any real function of x, y, z, let

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* The case of three variables, which includes the applications to physical problems, is alone considered here; although the analysis is equally applicable whatever be the number of variables.

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(\alpha\frac{dV}{dx}-\frac{1}{\alpha}\frac{dU}{dx}\right)^2+\left(\alpha\frac{dV}{dy}-\frac{1}{\alpha}\frac{dU}{dy}\right)^2\right.$$
$$\left.+\left(\alpha\frac{dV}{dz}-\frac{1}{\alpha}\frac{dU}{dz}\right)^2\right\}dxdydz \quad (b).$$

It is obvious that, although V may be assigned so as to make Q as great as we please, it is impossible to make the value of Q less than a certain limit, since we see at once that it cannot be negative. Hence Q, considered as depending on the arbitrary function V, is susceptible of a minimum value; and the calculus of variations will lead us to the assigning of V according to this condition.

Thus we have

$$\tfrac{1}{2}\delta Q=\iiint\left\{\left(\alpha\frac{dV}{dx}-\frac{1}{\alpha}\frac{dU}{dy}\right).\alpha\frac{d\delta V}{dx}+\left(\alpha\frac{dV}{dy}-\frac{1}{\alpha}\frac{dU}{dy}\right).\alpha\frac{d\delta V}{dy}\right.$$
$$\left.+\left(\alpha\frac{dV}{dz}-\frac{1}{\alpha}\frac{dU}{dz}\right).\alpha\frac{d\delta V}{dz}\right\}dxdydz.$$

Hence, by the ordinary process of integration by parts, the integrated terms vanishing at each limit,* we deduce

$$-\tfrac{1}{2}\delta Q=\iiint\delta V\left\{\frac{d}{dx}\left(\alpha^2\frac{dV}{dx}-\frac{dU}{dx}\right)+\frac{d}{dy}\left(\alpha^2\frac{dV}{dy}-\frac{dU}{dy}\right)\right.$$
$$\left.+\frac{d}{dz}\left(\alpha^2\frac{dV}{dz}-\frac{dU}{dz}\right)\right\}dxdydz.$$

But by a well-known theorem (proved in Pratt's *Mechanics*, and in the treatise on Attraction in Earnshaw's *Dynamics*), we have

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=-4\pi\rho.$$

Hence the preceding expression becomes

$$-\tfrac{1}{2}\delta Q=\iiint\delta V.\left\{\frac{d}{dx}\left(\alpha^2\frac{dV}{dx}\right)+\frac{d}{dy}\left(\alpha^2\frac{dV}{dy}\right)\right.$$
$$\left.+\frac{d}{dz}\left(\alpha^2\frac{dV}{dz}\right)+4\pi\rho\right\}dxdydz.$$

We have, therefore, for the condition that Q may be a maximum or minimum, the equation,

$$\frac{d}{dx}\left(\alpha^2\frac{dV}{dx}\right)+\frac{d}{dy}\left(\alpha^2\frac{dV}{dy}\right)+\frac{d}{dz}\left(\alpha^2\frac{dV}{dz}\right)=-4\pi\rho,$$

to be satisfied for all values of the variables.

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* All the functions of x, y, z contemplated in this paper are supposed to vanish for infinite values of the variables.

Now it is possible to assign V so that Q may be a minimum, and therefore there exists a function, V, which satisfies equation (A).

2. (*Demonstration*).—Let V be a solution of (A), and let V_1 be any different function of x, y, z, that is to say, any function such that $V_1 - V$, which we may denote by ϕ, does not vanish for all values of x, y, z. Let us consider the integral Q_1, obtained by substituting V_1 for V in the expression for Q. Since

$$\left(a\frac{dV_1}{dx}-\frac{1}{a}\frac{dU}{dx}\right)^2=\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right)^2+2\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right)a\frac{d\phi}{dx}+a^2\frac{d\phi^2}{dx^2},$$

we have

$$Q_1=Q+2\iiint\left\{\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right)a\frac{d\phi}{dx}+\left(a\frac{dV}{dy}-\frac{1}{a}\frac{dU}{dy}\right)a\frac{d\phi}{dy}\right.$$
$$\left.+\left(a\frac{dV}{dz}-\frac{1}{a}\frac{d\phi}{dz}\right)a\frac{d\phi}{dz}\right\}dxdydz$$
$$+\iiint a^2\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dxdydz.$$

Now, by integration by parts, we find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right)a\frac{d\phi}{dx}\cdot dxdydz$$
$$=-\iiint\phi\cdot\frac{d}{dx}\left(a^2\frac{dV}{dx}-\frac{dU}{dx}\right)dxdydz,$$

the integrated term vanishing at each limit. Applying this and similar processes with reference to y and z, we find an expression for the second term of Q_1, which, on account of equation (A), vanishes. Hence

$$Q_1=Q+\iiint a^2\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dxdydz \qquad (c),$$

which shows that Q_1 is greater than Q. Now the only peculiarity of Q is, that V, from which it is obtained, satisfies the equation (A), and therefore V_1 cannot be a solution of (A). Hence no function different from V can be a solution of (A).

The analysis given above, especially when interpreted in various cases of abrupt variations in the value of a, and of infinite or evanescent values, through finite spaces, possesses very important applications in the theories of heat, electricity, magnetism, and hydrodynamics, which may form the subject of future communications.

EDINBARNET, DUMBARTONSHIRE, *Oct.* 9, 1847.

ADDITION TO A FRENCH TRANSLATION OF THE PRECEDING.

[From Liouville's *Journal de Mathématiques*, 1847.]

207. Dans les applications qui présentent le plus d'intérêt, il faut considérer des transitions subites dans la valeur de a. Par exemple, si a a une valeur constante dans tout l'espace extérieur à une surface fermée S, dans l'intérieur de laquelle a est infinie, notre analyse convient au cas d'un corps conducteur S soumis à l'influence d'une masse électrique donnée $(\iiint \rho\, dxdydz)$, et cette application ne présente aucune difficulté. On en tire, en effet, les démonstrations données par Green, que la solution analytique du problème de la distribution d'électricité dans ces circonstances est possible et qu'elle est unique.

Dans une application à l'hydrodynamique, ou à un certain problème de magnétisme, il faut considérer un espace dans lequel la valeur de a soit zéro. L'interprétation du resultat ne présente aucune difficulté, mais il est plus difficile de bien comprendre comment la démonstration telle que je l'ai donnée plus haut se prête à ce cas. En essayant de l'expliquer nettement, j'ai trouvé une démonstration directe du théorème suivant, qui renferme le résultat dont il s'agit:

"Il est possible de trouver une fonction V qui s'évanouisse pour les valeurs infiniment grandes des variables x, y, z, et satisfasse à l'équation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0,$$

pour tous les points extérieurs à une surface fermée S, avec cette condition

$$\frac{dV}{dn}=F,$$

dans laquelle F est une fonction arbitraire des coordonnées d'un point sur la surface S, et dn est l'élément d'une normale extérieure à la surface en ce point."

Pour le démontrer, considérons l'intégrale

$$\iiint\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right]dxdydz=Q,$$

relative à l'espace extérieur à S. Parmi toutes les fonctions V qui vérifient la condition

$$\iint VFdS = A,$$

où A est une quantité quelconque, il y en a une pour laquelle l'intégrale Q est un minimum. Une fonction V, ainsi déterminée, satisfait aux équations

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0,$$

$$\frac{dV}{dn}=cF$$

(où c est une constante), comme on s'en assure par le calcul des variations. Suivant les valeurs de A, c aura des valeurs proportionnelles; on peut prendre A telle que $c=1$. De là on conclut le théorème énoncé. Il serait facile d'ajouter une démonstration, que la solution du problème de la détermination de V sous ces conditions est unique.*

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* [Provided S is a simply continuous surface. If S be a multiply continuous surface, as, for instance, the inner boundary of an endless tube (a finite tube with its ends united, so as to constitute a circuit), we may add to V the velocity-potential of a liquid moving through it irrotationally (Thomson and Tait's *Natural Philosophy*, §§ 184...190; Thomson, *Vortex Motion*, §§ 54...58) without violating the conditions prescribed in the text. Compare above, § 200, footnote.]

XIV. ELECTRIC IMAGES.

EXTRAIT D'UNE LETTRE DE M. WILLIAM THOMSON À M. LIOUVILLE.

[From Liouville's *Journal de Mathématiques*, 1845.]

"CAMBRIDGE, 8 *Octobre* 1845.

208. "... Pendant mon séjour à Paris, je vous ai parlé du principe des *images* pour la solution de quelques problèmes relatifs à la distribution de l'électricité. Il y a une foule de problèmes auxquels je ne pensais pas alors, et où j'ai trouvé plus tard qu'on peut l'appliquer. Par exemple, on parvient ainsi à exprimer algébriquement la distribution d'électricité sur deux plans conducteurs qui se coupent sous un angle $\frac{\pi}{i}$, quand un point électrique est posé dans l'espace entre les deux plans. (L'idée est analogue à celle du *kaleidoscope* de Brewster.) Quand il y a trois plans qui se coupent perpendiculairement, ou quand il y a un plan qui coupe perpendiculairement deux plans qui se coupent sous un angle $\frac{\pi}{i}$, on peut également trouver la distribution sous l'influence d'un point électrique donné. On peut aussi exprimer très-facilement la distribution sur les parois intérieures d'un parallélipipède rectangulaire creux, soumis à l'influence d'un point électrique posé en dedans, en se servant des intégrales définies.

"Soient C le centre d'une sphère S; Q, Q' deux points pris sur un même rayon CA et sur son prolongement, de telle manière que

$$CQ . CQ' = CA^2;$$

et P un point quelconque sur la surface S. On a, comme on sait,

$$\frac{PQ}{PQ'} = \frac{AQ}{AQ'}.$$

On peut, à cause de ce théorème, appeler Q et Q' *points réciproques relatifs à la sphère S*, dont chacun est *l'image* de l'autre dans la sphère. Suivant cette définition, l'image d'une ligne ou surface sera le lieu des images de points pris sur cette ligne ou surface. Ainsi, on trouve que l'image d'un plan ou d'une sphère est toujours une sphère (le plan étant compris sous cette désignation). Les images de deux sphères se coupent sous le même angle, réel ou imaginaire, que les surfaces données.

"Soient Q, Q' deux points réciproques, relativement à une sphère S, et q, q', s leurs images et l'image de la sphère S dans une autre sphère donnée. Les points q, q' seront réciproques relativement à la sphère s.

209. "A l'aide de ces théorèmes, je parviens facilement déterminer les images successives d'un point quelconque (qui n'est pas nécessairement dans la ligne qui passe par leurs centres), dans deux sphères qui se coupent sous un angle donné. Quand cet angle est imaginaire, je parviens ainsi à exprimer la distribution de l'électricité sur les deux sphères, sous l'influence d'un point quelconque, chargé d'électricité, au moyen des séries de M. Poisson (qui convergent comme des séries géométriques). Quand l'angle d'intersection est réel et compris dans l'expression $\frac{\pi}{i}$, on parvient ainsi à exprimer algébriquement la distribution d'une quantité donnée d'électricité sur la surface extérieure des sphères, qui n'est soumise à aucune influence ou qui l'est à celle d'un point donné. S'il y a trois surfaces sphériques qui se coupent perpendiculairement, on exprime algébriquement, par les mêmes principes, la distribution sur la surface extérieure. Je parviens aussi à déterminer les températures stationnaires dans l'intérieur d'une lentille dont les deux surfaces se coupent sous un angle $\frac{\pi}{i}$, la température de chaque point de ces surfaces étant donnée.

210. "Si l'on veut déterminer la distribution d'électricité sur une surface donnée S, sous l'influence d'un point quelconque Q, on réduit, par les mêmes principes, le problème à la détermination de la distribution, sans aucune influence, sur l'image de S dans une sphère décrite du centre Q, avec un rayon quelconque. Une application générale de ce théorème conduit à une démonstration rigoureuse du théorème de M. Gauss, qu'on peut produire, au moyen d'une distribution déterminée de matière sur

une surface fermée quelconque, une valeur donnée du potentiel à chaque point de la surface. Il y a aussi beaucoup d'applications spéciales [see below, §§ 218...220] qu'on peut faire de ce théorème aux cas dans lesquels S est une sphère, un disque circulaire, ou un segment d'une surface sphérique fait par un plan. J'en ai aussi déduit une démonstration géométrique du théorème que vous avez publié dans le numéro d'avril 1845 de votre Journal (*voir* page 137), dont voici l'expression analytique * * *" [see above, XI. §§ 167, 186].

EXTRAITS DE DEUX LETTRES ADDRESSÉES À M. LIOUVILLE, PAR M. WILLIAM THOMSON.

[From Liouville's *Journal de Mathématiques*, 1847.]

"CAMBRIDGE, 26 *juin* 1846.

211. "... Les recherches sur lesquelles je vous ai écrit, le 8 octobre 1845, m'ont conduit à l'emploi d'un système nouveau de coordonnées orthogonales très-commode dans quelques problèmes des théories de la chaleur et de l'électricité. Les surfaces coordonnées dans ce système sont les surfaces engendrées par la rotation, autour d'un axe convenable, d'un système de coordonnées curvilignes dans un plan, et les plans méridiens. En effet, soit M un plan méridien quelconque; les coordonnées d'un point P dans ce plan sont deux cercles qui se coupent à angle droit en ce point, et dont le premier passe par deux points fixes A, A', dans l'axe de révolution $X'X$, tandis que le second est la courbe orthogonale de la série entière des cercles qui passent par les points A, A'. On démontre facilement que cette courbe est un cercle qui passe par deux points imaginaires B, B', dans la droite $Y'OY$ perpendiculaire à $X'OX$, à des distances aux deux côtés de O dont chacune est égale à $a\sqrt{-1}$, a étant la valeur des distances égales $A'O$, OA. En effet, la première série est exprimée par l'équation

$$x^2+y^2-2uy=a^2, \tag{1}$$

u étant un paramètre variable, et l'on en déduit

$$x^2+y^2-2vx=-a^2, \tag{2}$$

pour l'équation de la courbe orthogonale.

212. “Posons

$$u = a \cot\theta, \quad v = a\sqrt{-1}\,.\cot\psi;$$

θ sera l'angle que la tangente du cercle (1), au point A ou A', fait avec l'axe $X'X$, et ψ sera l'angle imaginaire que la tangente du cercle (2), au point B ou B', fait avec $Y'Y$. Pour avoir la série entière des cercles (1), il faudrait donner à u toutes les valeurs réelles de $-\infty$ à ∞, ou à θ toutes les valeurs de 0 à π; et, pour la série (2), il faudrait donner à v toutes les valeurs de a à ∞, et de $-\infty$ à $-a$. On peut considérer un point P comme déterminé sans ambiguïté par les coordonnées θ, ψ (en prenant $\theta+\pi$ au lieu de θ pour l'autre point d'intersection des mêmes cercles). Les équations de transformation, entre les coordonnées (x, y) et (θ, ψ) d'un même point P, sont

$$x^2+y^2-2ay\cot\theta = a^2, \tag{3}$$

$$x^2+y^2-2ax\cot\psi\sqrt{-1} = -a^2. \tag{4}$$

On en déduit

$$x = -a\,\frac{\sin\psi\sqrt{-1}}{\cos\psi-\cos\theta},$$

$$y = a\,\frac{\sin\theta}{\cos\psi-\cos\theta},$$

$$x^2+y^2 = a^2\,\frac{\cos\psi+\cos\theta}{\cos\psi-\cos\theta},$$

Dans les applications physiques, il s'agit d'exprimer la distance Δ, entre deux points P, P' en fonction des nouvelles coordonnées. On trouve facilement, à l'aide des formules données ci-dessus, dans le cas de P et P' dans un même plan méridien M,

$$\Delta^2 = 2a^2\,\frac{\cos(\psi-\psi')-\cos(\theta-\theta')}{(\cos\psi-\cos\theta)(\cos\psi'-\cos\theta')}.$$

Pour le trois coordonnées d'un point dans l'espace, je prends θ, ψ qui fixent sa position dans un plan méridien, et l'angle ϕ que ce plan fait avec un plan méridien fixe. Je trouve maintenant, pour la distance entre deux points quelconques P, P',

$$\Delta^2 = 2a^2\,\frac{\cos(\psi-\psi')-[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')]}{(\cos\psi-\cos\theta)(\cos\psi'-\cos\theta')}.$$

Pour éviter l'emploi de quantités imaginaires, je pose

$$2\cos\psi = r+\frac{1}{r}, \quad 2\cos\psi' = r'+\frac{1}{r'},$$

d'où l'on déduit

$$2\cos(\psi-\psi')=\frac{r}{r'}+\frac{r'}{r},$$

et l'expression précédente se réduit à

$$\Delta^2=a^2\frac{r^2-2rr'[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')]+r'^2}{(r^2-2r\cos\theta+1)(r'^2-2r'\cos\theta'+1)}.$$

A l'aide cette expression, on trouve

$$r\frac{d^2(s^{-1}vr)}{dr^2}+\frac{1}{\sin\theta}\frac{d\left[\sin\theta\dfrac{d(s^{-1}v)}{d\theta}\right]}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2(s^{-1}v)}{d\phi^2}=0,$$

où

$$s=(r^2-2r\cos\theta+1)^{\frac{1}{2}},$$

pour l'équation du mouvement uniforme de la chaleur exprimée par les coordonnées r, θ, ϕ.

"Les surfaces représentées par l'équation

$$r=\text{constante}$$

sont des sphères engendrées par la révolution d'une série de cercles autour de la droite qui contient leurs centres. Supposons que l'espace entre deux de ces sphères (quand chaque sphére est en dehors de l'autre, cet espace sera l'espace infini en dehors des deux sphéres), dont les équations sont

$$r=a,\quad r=a_1,$$

soit rempli d'un milieu solide homogène, que les températures de tous les points de chaque surface soient données, et qu'il s'agisse de déterminer la température stationnaire d'un point quelconque dans le solide; on résoudra ce problème avec beaucoup de facilité au moyen de l'analyse de Laplace, en employant les coordonnées que j'ai indiquées. Dans le cas particulier d'une température constante pour chaque sphère, on parvient, après quelques réductions, à trouver la solution que Poisson a donnée pour le problème correspondant de deux sphères électrisées.

213. "Il y a un système nouveau et très-remarquable de coordonnées, qu'on trouve en posant

$$r\cos\theta=\xi,\quad r\sin\theta\cos\phi=\eta,\quad r\sin\theta\sin\phi=\zeta,$$

r, θ, ϕ appartenant au système expliqué ci-dessus. Dans ce système (ξ, η, ζ), les surfaces coordonnées sont des sphères orthogonales qui passent par un point fixe, et qui touchent, par conséquent, trois plans orthogonaux menés par ce point. Je suis parvenu à considérer ces systèmes de coordonnées en cherchant les *images* des séries de surfaces des systèmes (polaire et rectangulaire) ordinaires, dans des sphères convenablement disposées.

"L'application du système (ξ, η, ζ) aux problèmes de physique, pour le cas de deux systèmes qui se touchent l'un l'autre, en donne les solutions avec beaucoup de facilité; mais il est plus simple de faire directement la recherche de ces coordonnées, que de les déduire du système (r, θ, ϕ). En effet, soient

$$x^2+y^2+z^2-\frac{x}{\xi}=0,$$

$$x^2+y^2+z^2-\frac{y}{\eta}=0,$$

$$x^2+y^2+z^2-\frac{z}{\zeta}=0,$$

les équations de trois sphères qui se coupent à un point P (elles se coupent aussi à l'origine O). Je prends ξ, η, ζ pour les coordonnées de ce point (il faudrait substituer $\frac{\xi-1}{a}, \frac{\eta}{a}, \frac{\zeta}{a}$ dans ces équations, au lieu de ξ, η, ζ pour retrouver les coordonnées ξ, η, ζ indiquées ci-dessus). De ces équations on tire

$$x^2+y^2+z^2=\frac{1}{\xi^2+\eta^2+\zeta^2},$$

$$x=\frac{\xi}{\xi^2+\eta^2+\zeta^2}, \quad y=\frac{\eta}{\xi^2+\eta^2+\zeta^2}, \quad z=\frac{\zeta}{\xi^2+\eta^2+\zeta^2},$$

$$(x-x')^2+(y-y')^2+(z-z')^2=\frac{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}{(\xi^2+\eta^2+\zeta^2)(\xi'^2+\eta'^2+\zeta'^2)},$$

et l'équation

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0$$

devient, pour les nouvelles coordonnées,

$$(a) \qquad \frac{d^2(\rho^{-1}v)}{d\xi^2}+\frac{d^2(\rho^{-1}v)}{d\eta^2}+\frac{d^2(\rho^{-1}v)}{d\zeta^2}=0,$$

où

$$\rho=(\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}.$$

Pour exemple de l'emploi qu'on peut faire de ce système de coordonnées, supposons que la température d'un point (a, η, ζ) est une fonction donnée $F(\eta, \zeta)$ des coordonnées η, ζ de sa position sur la sphère a, et que la température d'un point (a_1, η, ζ) est $F_1(\eta, \zeta)$, et qu'il s'agit de déterminer la température permanente d'un point quelconque P (ξ, η, ζ) dans l'espace entre les sphères a, a_1 (c'est-à-dire l'espace entier pour lequel ξ a une valeur intermédiaire à a et a_1), que nous supposerons rempli d'un solide homogène. Suivant la méthode de Fourier, en observant que les valeurs

$$\cos m\eta . \cos n\zeta . \varepsilon^{h\xi},$$
$$\cos m\eta . \sin n\zeta . \varepsilon^{h\xi},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

substituées pour $\rho^{-1}v$, sont des solutions particulières de l'équation (a), pourvu que $h^2 = m^2 + n^2$, je trouve, pour la solution du problème proposé,

$$(b)\left\{\begin{aligned} &v = \frac{\rho}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm\,dn \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta' d\zeta' \frac{\cos m(\eta-\eta')\cos n(\zeta-\zeta')}{\varepsilon^{h(a-a_1)} - \varepsilon^{-h(a-a_1)}} \\ &\times\left\{\begin{aligned} &\frac{F(\eta',\zeta')}{(a^2+\eta'^2+\zeta'^2)^{\frac{1}{2}}}\left[\varepsilon^{h(\xi-a_1)} - \varepsilon^{-h(\xi-a_1)}\right] \\ -&\frac{F_1(\eta',\zeta')}{(a_1^2+\eta'^2+\zeta'^2)^{\frac{1}{2}}}\left[\varepsilon^{h(\xi-a)} - \varepsilon^{-h(\xi-a)}\right] \end{aligned}\right\}, \end{aligned}\right.$$

où ε est la base des logarithmes népériens, et

$$h = (m^2 + n^2)^{\frac{1}{2}}.$$

214. "Comme exemple de l'usage de cette formule, je ferai

$$F(\eta, \zeta) = F_1(\eta, \zeta) = V,$$

V étant une constante. Pour la réduction de l'expression, dans ce cas, j'observe que

$$(c) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq \frac{\cos mp \cos nq}{(\lambda+p^2+q^2)^{\frac{1}{2}}} = 2\pi \frac{\varepsilon^{-(m^2+n^2)^{\frac{1}{2}}\lambda^{\frac{1}{2}}}}{(m^2+n^2)^{\frac{1}{2}}}$$

d'où l'on déduit

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta' d\zeta' \frac{\cos m(\eta-\eta')\cos((\zeta-\zeta')}{(a^2+\eta'^2+\zeta'^2)^{\frac{1}{2}}} = 2\pi \cos m\eta . \cos n\zeta . \frac{\varepsilon^{-ha}}{h},$$

et

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta' d\zeta' \frac{\cos m(\eta-\eta')\cos n((\zeta-\zeta')}{(a_1^2+\eta'^2+\zeta'^2)^{\frac{1}{2}}} = 2\pi \cos m\eta . \cos n\zeta . \frac{\varepsilon^{\mp ha_1}}{h}$$

le signe supérieur ou inférieur étant pris, dans la seconde expression, selon que a_1 est positif ou négatif (je prends a toujours positif et $> a_1$), Ces réductions faites, l'expression (b) se trouve réduite à

(I) $$v=\frac{V\rho}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dm\,dn\cos m\eta\,.\cos n\zeta\,.\frac{\varepsilon^{-(m^2+n^2)^{\frac{1}{2}}\xi}}{(m^2+n^2)^{\frac{1}{2}}}$$

et

(II.) $$\left\{\begin{array}{l} v=\dfrac{V\rho}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dm\,dn\cos m\eta\cos n\zeta \\ \times\dfrac{(\varepsilon^{-ha}-\varepsilon^{ha_1})\varepsilon^{h(\xi-a)}+(\varepsilon^{-ha}-\varepsilon^{ha})\varepsilon^{-h(\xi-a_1)}}{h[\varepsilon^{h(a-a_1)}-\varepsilon^{-h(a-a_1)}]}\end{array}\right.$$

suivant les deux cas. L'équation (I) se réduit à

$$v=V,$$

à cause de la valeur qu'on trouve pour l'intégrale définie qui y est contenue.*

215. "L'expression pour v, dans le second cas, se trouve réduite en série convergente, si l'on substitue pour

$$\frac{1}{\varepsilon^{h(a-a_1)}-\varepsilon^{-h(a-a_1)}}\text{ la série}$$

$$\varepsilon^{-h(a-a_1)}[1+\varepsilon^{-2h(a-a_1)}+\varepsilon^{-4h(a-a_1)}+\ldots],$$

et puis, pour chaque terme, sa valeur, suivant la formule citée dans le cas (I). On trouve ainsi

$$v=V\rho\left\{\begin{array}{l}\dfrac{1}{[(2a-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\dfrac{1}{[(\gamma+2a-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\dfrac{1}{[(2\gamma+2a-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\ldots. \\ -\dfrac{1}{[(\gamma-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}-\dfrac{1}{[(2\gamma-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}-\dfrac{1}{[(3\gamma-\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}-\ldots\ldots\ldots \\ +\dfrac{1}{[(\xi-2a_1)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\dfrac{1}{[(\gamma+\xi-2a_1)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\dfrac{1}{[(2\gamma+\xi-2a_1)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}+\ldots \\ -\dfrac{1}{[(\gamma+\xi)^2+\eta^2+\xi^2]^{\frac{1}{2}}}-\dfrac{1}{[(2\gamma+\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}-\dfrac{1}{[(3\gamma+\xi)^2+\eta^2+\zeta^2]^{\frac{1}{2}}}-\ldots\ldots\ldots\end{array}\right\},$$

où $$\gamma=2(a-a_1).$$

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* Les intégrales définies (*c*) et (I) sont des cas particuliers de deux intégrales multiples dont j'ai trouvé les valeurs en cherchant une démonstration de la formule (5), tome X de votre Journal, page 141. J'ai trouvé, [above, § 182, formula (V)], en effet,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\frac{dp_1dp_2\ldots dh_n\cos m_1p_1\cos m_2p_2\ldots}{(p_1^2+p_2^2+\ldots+p_n^2+u^2)^{\frac{n-1}{2}}}=\frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,\frac{\varepsilon^{-(m_1^2+m_2^2+\ldots)^{\frac{1}{2}}u}}{(m_1^2+m_2^2+\ldots)^{\frac{1}{2}}}$$

et

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\frac{dm_1dm_2\ldots\cos m_1x_1\cos m_2x_2\ldots\varepsilon^{-(m_1^2+m_2^2+\ldots)^{\frac{1}{2}}u}}{(m_1^2+m_2^2\ldots)^{\frac{1}{2}}}$$

$$=\frac{2^n\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{(n-1)(x_1^2+x_2^2+\ldots+x_n+u^2)^{\frac{n-1}{2}}}$$

d'où l'on déduit immédiatement les intégrales citées.

De cette expression on déduit facilement la distribution d'électricité sur deux sphères qui se touchent.

216. "Le cas (I) correspond à deux sphères dont l'une, (a), est en dedans de l'autre, (a_1). Dans le cas (II), le solide considéré remplit l'espace entier en dehors des deux sphères, et la température est zéro à une distance infinie.

217. "Il y a une interprétation pour le nouveau système de coordonnées (r, θ) dans un plan, qui est très-simple. En effet, soient A, A' deux points fixes, et P un point quelconque dont il s'agit d'exprimer la position. Cela peut se faire au moyen de l'angle APA', que j'appelle θ, et de la raison r de AP à AP'. Quand θ a une valeur constante, le lieu de P est un cercle qui passe par les points A, A'; et quand r a une valeur constante, le lieu de P est un cercle, dont le centre est dans le prolongement de AA', d'un côté ou de l'autre, suivant que cette valeur est plus grande ou plus petite que l'unité, et qui a la propriété de couper à angle droit tout cercle décrit par les points A, A'.

"Posons maintenant, pour expliquer le second système,

$$r \cos \theta = \xi, \quad r \sin \theta = \eta.$$

Le lieu de P, quand ξ a une valeur constante, sera tel que, si l'on mène, de A, AD perpendiculaire à $A'P$, la raison $DP \div AP$ sera constante, et l'on trouve ainsi que ce lieu est un cercle qui touche en A' une droite perpendiculaire à $A'A$; et l'on trouve semblablement que le lieu de P, quand η a une valeur constante, est un cercle qui touche $A'A$ au point A'."

"KNOCK, le 16 septembre 1846.

218. "... Depuis que je vous ai écrit la dernière fois, j'ai considéré le problème de la distribution d'électricité sur le segment d'une couche sphérique infiniment mince, fait par un plan, ce corps étant composé de matière conductrice, et j'ai trouvé, en expression finie, la solution complète, en supposant que le corps possède une quantité donnée d'électricité et que la distribution se fait sous l'influence de masses électriques données. J'avais l'intention de rédiger de suite pour vous un petit Mémoire sur ces recherches, mais j'ai rencontré quelque difficulté dans l'exposition de la méthode suivie, et comme je suis à présent très-occupé (les cours à Glasgow commencent le 1er novembre, et il me faudra beaucoup de préparation), il me faut différer cette tâche.* Je me bornerai pour le moment aux énoncés de quelques-uns des résultats.

219. "Soit *S* le corps conducteur sur lequel il s'agit de déterminer la distribution. Pour premier cas, soit *Q* un point en dehors de *S*, sur la même surface sphérique dont *S* fait partie, et supposons que *S* soit mis en communication avec le sol par un fil conducteur infiniment mince (ainsi le potentiel dans *S* sera toujours zéro, quels que soient les corps électrisés qui en soient voisins). Il s'agit de déterminer la distribution d'électricité sur *S* sous l'influence d'une quantité donnée d'électricité négative *Q*, concentrée au point *Q*. Je démontre que l'intensité d'électricité a la même valeur aux points voisins des deux côtés de la couche *S*, et, en dénotant par σ cette valeur, pour un point quelconque *P* de *S*, je trouve

$$\sigma = \frac{Q}{2\pi^2} \cdot \frac{(s^2 - a^2)^{\frac{1}{2}}}{(a^2 - r^2)^{\frac{1}{2}} . \Delta^2},$$

où a, s et r sont les distances du bord de *S*, du point *Q* et du point *P*, à un point *C* de *S* qu'on peut appeler son centre, et Δ est la distance entre *Q* et *P*. Il est remarquable que cette expression ne contient pas le rayon de la sphère dont *S* fait partie. En supposant que ce rayon soit infini, on a l'expression pour la distribution d'électricité sur un disque circulaire, sous l'influence d'un point dans son plan, qui est, en effet, la même que celle que Green a donnée pour ce cas.

220. "Pour trouver la distribution dans le cas de *S* isolé et électrisé, je remarque que, si la quantité d'électricité sur *S* est telle que le potentiel qui en résulte a une valeur donnée *V*, la distribution sur *S* sera la même que celle qui aurait lieu si *S* était situé dans l'intérieur d'une couche électrique qui produit le potentiel $-V$, *S* étant dans l'état d'un corps qui n'est pas isolé. On peut prendre pour cette couche une sphère concentrique avec celle dont *S* fait partie; en supposant l'excès du rayon de la première sphère sur le rayon de la seconde infiniment petit, on réduit le problème à la détermination de la distribution sur *S*, sous l'influence d'une distribution donnée d'électricité sur la sphère dont *S* fait partie, ce corps *S* n'étant

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* It has, in fact, been delayed till December 1868 and January 1869. See XV. below.

pas isolé. Ainsi, par intégration, je déduis du résultat donné ci-dessus les expressions

$$\sigma=\frac{V}{2\pi^2 f}\left[\left(\frac{f^2-a^2}{a^2-r^2}\right)^{\frac{1}{2}}-\text{arc tang}\left(\frac{f^2-a^2}{a^2-r^2}\right)\right]+\frac{V}{2\pi f},$$

$$\sigma'=\frac{V}{2\pi^2 f}\left[\left(\frac{f^2-a^2}{a^2-r^2}\right)^{\frac{1}{2}}-\text{arc tang}\left(\frac{f^2-a^2}{a^2-r^2}\right)^{\frac{1}{2}}\right]$$

(où f est le diamétre de la sphère dont S fait partie), pour les intensités sur les deux côtés, convexe et concave, de S en un point P."

NOTE AU SUJET DE L'ARTICLE PRÉCÉDENT;

PAR J. LIOUVILLE.

221.[*] La Lettre de M. Thomson m'a suggéré quelques remarques que je crois devoir présenter ici, parce qu'elles montreront, ce me semble, plus clairement encore toute l'importance du travail dont le jeune géomètre de Glasgow nous a donné un extrait rapide.

Nous résoudrons d'abord le problème suivant :

Problème.—Soient $x, y, \ldots, z$ et $\xi, \eta, \ldots, \zeta$ deux groupes contenant un nombre égal ou inégal de variables, les premières $x, y, \ldots, z$ indépendantes, les autres $\xi, \eta, \ldots, \zeta$ fonctions des premières, en sorte que

$$\xi=f(x, y, \ldots, z),\quad \eta=F(x, y, \ldots, z), \ldots,\quad \zeta=\phi(x, y, \ldots, z);$$

soit encore $$p=\psi(x, y, \ldots, z).$$

Désignons d'ailleurs par $\xi', \eta', \ldots, \zeta', p'$ ce que deviennent les fonctions $\xi, \eta, \ldots, \zeta, p$, quand on y remplace $x, y, \ldots, z$ par $x', y', \ldots, z'$. Cela posé, on demande de déterminer les fonctions $f, F, \ldots, \phi, \psi$, de manière à avoir généralement

$$(\xi'-\xi)^2+(\eta'-\eta)^2+\ldots+(\zeta'-\zeta)^2=\frac{(x'-x)^2+(y'-y)^2+\ldots+(z'-z)^2}{p^2p'^2}.$$

Pour fixer les idées, nous nous bornerons au cas de trois variables x, y, z, et de trois variables ξ, η, ζ; et la question sera de vérifier l'équation

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[*] [The original numbering of M. Liouville's sections has been altered by the addition of 220, for more convenient reference in the present volume.]

$$(1)\qquad (\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{p^2p'^2}.$$

La même méthode réussirait pour deux groupes $x, y, \ldots, z$ et $\xi, \eta, \ldots, \zeta$ quelconques. Il n'y aurait de changement que dans quelques détails, et seulement si le nombre des variables était différent dans les deux groupes. Au surplus, nous n'aurons besoin plus tard que du cas où ce nombre est le même de part et d'autre, et ne surpasse pas trois, ce qui nous permettra d'interpréter géométriquement les résultats de notre analyse.

Donnons à x', y', z' des valeurs particulières x_0, y_0, z_0 à volonté, et représentons par p_0, ξ_0, η_0, ζ_0 les valeurs correspondantes de p', ξ', η', ζ'. L'équation (1) nous donnera

$$p^2=\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{p_0^2[(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2]}.$$

Mais, pour plus de simplicité, nous mettrons partout $\xi+\xi_0$, $\eta+\eta_0$, $\zeta+\zeta_0$, $x+x_0$, $y+y_0$, $z+z_0$, au lieu de ξ, η, ζ x, y, z, et de même $\xi'+\xi_0$, $x'+x_0$, etc., au lieu de ξ', x', etc., ce qui ne change rien aux différences $\xi'-\xi$, $x'-x$, etc. La valeur de p^2 deviendra

$$p^2=\frac{x^2+y^2+z^2}{p_0^2(\xi^2+\eta^2+\zeta^2)},$$

et l'équation (1) subsistera telle qu'elle est.

En faisant

$$x^2+y^2+z^2=r^2,\quad \xi^2+\eta^2+\zeta^2=\rho^2,$$
$$x'^2+y'^2+z'^2=r'^2,\quad \xi'^2+\eta'^2+\zeta'^2=\rho'^2,$$

on aura

$$p^2=\frac{r^2}{p_0^2\rho^2},\quad p'^2=\frac{r'^2}{p_0^2\rho'^2},$$

et en portant ces valeurs dans l'équation (1), on trouvera aisément

$$\frac{1}{\rho^2}+\frac{1}{\rho'^2}-2\left(\frac{\xi}{\rho^2}\frac{\xi'}{\rho'^2}+\frac{\eta}{\rho^2}\frac{\eta'}{\rho'^2}+\frac{\zeta}{\rho^2}\frac{\zeta'}{\rho'^2}\right)$$
$$=p_0^4\left[\frac{1}{r^2}+\frac{1}{r'^2}-2\left(\frac{x}{r^2}\frac{x'}{r'^2}+\frac{y}{r^2}\frac{y'}{r'^2}+\frac{z}{r^2}\frac{z'}{r'^2}\right)\right].$$

Maintenant donnons à x', y', z' quatre systèmes de valeurs connues à volonté, à chacun desquels répondront des valeurs déterminées de r', ξ', η', ζ', ρ', et nous aurons ainsi quatre équations du premier degré qui fourniront les valeurs de

$$\frac{\xi}{\rho^2},\quad \frac{\eta}{\rho^2},\quad \frac{\zeta}{\rho^2},\quad \frac{1}{\rho^2},$$

considérées comme quatre inconnues, en fonction linéaire de

$$\frac{x}{r^2},\ \frac{y}{r^2},\ \frac{z}{r^2},\ \frac{1}{r^2}.$$

En désignant donc par A, B, C, D des constantes, et par P, Q, R, S des polynômes du premier degré en x, y, z, ces valeurs seront de la forme

$$\frac{\xi}{\rho^2}=A+\frac{P}{r^2},\quad \frac{\eta}{\rho^2}=B+\frac{Q}{r^2},\quad \frac{\zeta}{\rho^2}=C+\frac{R}{r^2},\quad \frac{1}{\rho^2}=D+\frac{S}{r^2}.$$

En faisant la somme des carrés des trois premières, on trouve une valeur de $\frac{1}{\rho^2}$ qui doit être égale à celle que donne la quatrième équation. Ainsi les deux fonctions

$$D+\frac{S}{r^2}$$

et $$A^2+B^2+C^2+\frac{2(AP+BQ+CR)}{r^2}+\frac{P^2+Q^2+R^2}{r^4}$$

doivent être égales. Mais la première devient une fonction entière quand on la multiplie par r^2. Il faut donc que la seconde le devienne aussi, et que, par conséquent, $P^2+Q^2+R^2$ soit également divisible par r^2. Le quotient ne peut évidemment être qu'une constante, puisque le numérateur et le dénominateur sont du même degré. Soit m^2 cette constante, et

$$P^2+Q^2+R^2=m^2r^2=m^2(x^2+y^2+z^2).$$

P, Q, R étant des polynômes du premier degré, je fais

$$\begin{aligned}P&=m(ax+by+cz+g),\\ Q&=m(a'x+b'y+c'z+g'),\\ R&=m(a''x+b''y+c''z+g''),\end{aligned}$$

et j'en conclus par la comparaison des deux membres, d'une part,

$$\begin{aligned}&a^2+a'^2+a''^2=1, && ab+a'b'+a''b''=0,\\ &b^2+b'^2+b''^2=1, && ac+a'c'+a''c''=0,\\ &c^2+c'^2+c''^2=1, && bc+b'c'+b''c''=0,\end{aligned}$$

équations d'où résultent, comme on sait, les équations inverses

$$\begin{aligned}&a^2+b^2+c^2=1, && aa'+bb'+cc'=0,\\ &a'^2+b'^2+c'^2=1, && aa''+bb''+cc''=0,\\ &a''^2+b''^2+c''^2=1, && a'a''+b'b''+c'c''=0;\end{aligned}$$

et, d'autre part,

$$\begin{aligned}&ag+a'g'+a''g''=0, && cg+c'g'+c''g''=0,\\ &bg+b'g'+b''g''=0, && g^2+g'^2+g''^2=0.\end{aligned}$$

Si nous admettions que g, g', g'' sont des constantes réelles, l'équation $g^2+g'^2+g''^2=0$ nous donnerait $g=0$, $g'=0$, $g''=0$. Mais, dans tous les cas, on arrivera au même résultat à l'aide des trois précédentes, en ayant égard aux équations de condition entre a, b, c, etc. Pour prouver, par exemple, que $g=0$, il suffira d'ajouter entre elles les trois équations dont nous parlons après les avoir multipliées par les facteurs respectifs a, b, c. Il nous reste donc

$$P=m(ax+by+cz),$$
$$Q=m(a'x+b'y+c'z),$$
$$R=m(a''x+b''y+c''z),$$

a, b, c, etc., satisfaisant aux équations de condition ci-dessus, les mêmes qu'on rencontre dans la transformation de coordonnées rectangulaires en d'autres rectangulaires aussi. Et comme les équations

$$\frac{\xi}{\rho^2}=A+\frac{P}{r^2},\quad \frac{\eta}{\rho^2}=B+\frac{Q}{r^2},\quad \frac{\zeta}{\rho^2}=C+\frac{R}{r^2}$$

donnent

$$\frac{1}{\rho^2}=\left(A+\frac{P}{r^2}\right)^2+\left(B+\frac{Q}{r^2}\right)^2+\left(C+\frac{R}{r^2}\right)^2,$$

on en conclut les formules suivantes :

$$\xi=\frac{A+\dfrac{P}{r^2}}{\left(A+\dfrac{P}{r^2}\right)^2+\left(B+\dfrac{Q}{r^2}\right)^2+\left(C+\dfrac{R}{r^2}\right)^2},$$

$$\eta=\frac{B+\dfrac{Q}{r^2}}{\left(A+\dfrac{P}{r^2}\right)^2+\left(B+\dfrac{Q}{r^2}\right)^2+\left(C+\dfrac{R}{r^2}\right)^2},$$

$$\zeta=\frac{C+\dfrac{R}{r^2}}{\left(A+\dfrac{P}{r^2}\right)^2+\left(B+\dfrac{Q}{r^2}\right)^2+\left(C+\dfrac{R}{r^2}\right)^2}.$$

Mais il faut à présent rétablir $\xi-\xi_0$, $\eta-\eta_0$, $\zeta-\zeta_0$ au lieu de ξ, η, ζ et $x-x_0$, $y-y_0$, $z-z_0$ au lieu de x, y, z. Ce changement fait, on aura les formules les plus générales qui puissent satisfaire à l'équation (1). Nous avons donc le théorème suivant :

Les formules générales qui peuvent satisfaire à l'équation (1) s'obtiendront en posant d'abord

$$\mathrm{X}=a\,(x-x_0)+b\,(y-y_0)+c\,(z-z_0),$$
$$\mathrm{Y}=a'(x-x_0)+b'(y-y_0)+c'(z-z_0),$$
$$\mathrm{Z}=a''(x-x_0)+b''(y-y_0)+c''(z-z_0),$$

les coefficients a, b, etc., vérifiant les équations de condition

$$a^2+a'^2+a''^2=1,\quad ab+a'b'+a''b''=0,$$
$$b^2+b'^2+b''^2=1,\quad ac+a'c'+a''c''=0,$$
$$c^2+c'^2+c''^2=1,\quad bc+b'c'+b''c''=0,$$

puis prenant

$$u=A+\frac{m\mathrm{X}}{\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2},\quad v=B+\frac{m\mathrm{Y}}{\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2},\quad w=C+\frac{m\mathrm{Z}}{\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2},$$

et enfin

$$\xi-\xi_0=\frac{u}{u^2+v^2+w^2},\quad \eta-\eta_0=\frac{v}{u^2+v^2+w^2},\quad \zeta-\zeta_0=\frac{w}{u^2+v^2+w^2},$$

Réciproquement, on peut démontrer que l'equation (1) est satisfaite de cette manière, et trouver la valeur de p qui convient.

D'abord, des trois dernières formules on conclut facilement

$$(\xi'-\xi)^2+(\eta'-\eta)^2+\zeta'-\zeta)^2=\frac{(u'-u)^2+(v'-v)^2+(w'-w)^2}{(u^2+v^2+w^2)(u'^2+v'^2+w'^2)}:$$

les trois précédentes donnent de même

$$(u'-u)^2+(v'-v)^2+(w'-w)^2=m^2\frac{(\mathrm{X}'-\mathrm{X})^2+(\mathrm{Y}'-\mathrm{Y})^2+(\mathrm{Z}'-\mathrm{Z})^2}{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)(\mathrm{X}'^2+\mathrm{Y}'^2+\mathrm{Z}'^2)};$$

enfin, à cause des équations de condition entre a, b, etc., on trouve

$$(\mathrm{X}'-\mathrm{X})^2+(\mathrm{Y}'-\mathrm{Y})^2+(\mathrm{Z}'-\mathrm{Z})^2=(x'-x)^2+(y'-y)^2+(z'-z)^2.$$

Il vient donc, en effet,

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{p^2p'^2},$$

la valeur de p^2 étant

$$p^2=\frac{(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)(u^2+v^2+w^2)}{m},$$

valeur qu'on pourra aisément exprimer en x, y, z, en observant que le produit $(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)(u^2+v^2+w^2)$ est égal à

$$(A^2+B^2+C^2)(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)+2Am\mathrm{X}+2Bm\mathrm{Y}+2Cm\mathrm{Z}+m^2,$$

et que X, Y, Z sont connus en fonction de x, y, z. La valeur qu'on trouvera ainsi peut se mettre sous la forme

$$mp^2=(A^2+B^2+C^2)[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2],$$

x_1, y_1, z_1 étant des constantes dont voici les valeurs :

$$x_1=x_0-\frac{m(Aa+Ba'+Ca'')}{A^2+B^2+C^2},$$

$$y_1=y_0-\frac{m(Ab+Bb'+Cb'')}{A^2+B^2+C^2},$$

$$z_1=z_0-\frac{m(Ac+Bc'+Cc'')}{A^2+B^2+C^2}.$$

Si donc nous regardons plus tard x, y, z comme étant les coordonnées rectangulaires d'un point quelconque, on voit que la quantité p sera proportionnelle à la distance de ce point (x, y, z) à un point fixe (x_1, y_1, z_1). Il est aisé aussi de s'assurer que

$$\frac{d^2\frac{1}{p}}{dx^2}+\frac{d^2\frac{1}{p}}{dy^2}+\frac{d^2\frac{1}{p}}{dz^2}=0.$$

222. Pour avoir explicitement ξ, η, ζ en x, y, z, il suffira de remplacer u, v, w, X, Y, Z par leurs valeurs. La première substitution fournit

$$\xi-\xi_0=\frac{A(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)+m\mathrm{X}}{(A^2+B^2+C^2)(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)+2Am\mathrm{X}+2Bm\mathrm{Y}+2Cm\mathrm{Z}+m^2}.$$

Le dénominateur est précisément la valeur de mp^2 dont on vient de donner l'expression en x, y, z, savoir,

$$mp^2=(A^2+A^2+C^2)[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2].$$

Il ne reste donc plus qu'à chercher le numérateur. Le calcul deviendra d'ailleurs fort simple si l'on retranche des deux membres la quantité

$$\frac{A}{A^2+B^2+C^2},$$

car alors le second membre pourra se réduire à une fraction ayant pour numérateur un polynôme du premier degré en X, Y, Z, et, par conséquent aussi, en x, y, z. En désignant donc par X un tel polynôme, et posant, pour abréger,

$$\xi_0+\frac{A}{A^2+B^2+C^2}=\xi^0,$$

on pourra écrire $$\xi-\xi^0=\frac{X}{p^2},$$

et de même $\eta-\eta^0=\frac{Y}{p^2}$, $\zeta-\zeta^0=\frac{Z}{p^2}$,

η^0, ζ^0 étant des constantes, et Y, Z des fonctions linéaires de x, y, z. Les polynômes X, Y, Z s'obtiendraient sans peine par ce qu'on vient de dire; mais on les trouve sous une forme plus commode en opérant comme il suit. Il est aisé de voir qu'en attribuant une valeur infinie à une ou plusieurs des quantités x, y, z, ou, si l'on veut, en faisant

$$x^2+y^2+z^2=\infty,$$

on a $\xi=\xi^0$, $\eta=\eta^0$, $\zeta=\zeta^0$, $\frac{p^2}{x^2+y^2+z^2}=\frac{A^2+B^2+C^2}{m}$.

Si donc on introduit cette hypothèse de $x^2+y^2+z^2=\infty$ dans l'équation générale

$$(\xi'-\xi)^2+(\eta'-\eta)^2+\zeta'-\zeta)^2=\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{p^2p'^2},$$

il viendra

$$(\xi'-\xi^0)^2+(\eta'-\eta^0)^2+(\zeta'-\zeta^0)^2=\frac{m}{(A^2+B^2+C^2p'^2},$$

d'où, en effaçant les accents,

$$(\xi-\xi^0)^2+(\eta-\eta^0)^2+(\zeta-\zeta^0)^2=\frac{m}{(A^2+B^2+C^2p^2}.$$

Mais, d'un autre côté,

$$(\xi-\xi^0)^2+(\eta-\eta^0)^2+(\zeta-\zeta^0)^2=\frac{X^2+Y^2+Z^2}{p^4};$$

donc

$$X^2+Y^2+Z^2=\frac{mp^2}{A^2+B^2+C^2},$$

c'est-à-dire

$$X^2+Y^2+Z^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2.$$

De là, par un calcul tout semblable à celui qu'on a effectué dans le numéro précédent pour l'équation

$$P^2+Q^2+R^2=m^2(x^2+y^2+z^2),$$

on conclut qu'en représentant par α, β, γ, α', etc., des constantes assujetties aux équations de condition

$$\alpha^2+\alpha'^2+\alpha''^2=1,\quad \alpha\beta+\alpha'\beta'+\alpha''\beta''=0,$$
$$\beta^2+\beta'^2+\beta''^2=1,\quad \alpha\gamma+\alpha'\gamma'+\alpha''\gamma''=0,$$
$$\gamma^2+\gamma'^2+\gamma''^2=1,\quad \beta\gamma+\beta'\gamma'+\beta''\gamma''=0,$$

du même genre que celles entre a, b, etc., on devra prendre

$$X = \alpha\,(x-x_1)+\beta\,(y-y_1)+\gamma\,(z-z_1),$$
$$Y = \alpha'(x-x_1)+\beta'(y-y_1)+\gamma'(z-z_1),$$
$$Z = \alpha''(x-x_1)+\beta''(y-y_1)+\gamma''(z-z_1).$$

Et, réciproquement, il est facile de vérifier qu'en adoptant ces valeurs de X, Y, Z, les formules

$$\xi-\xi^{\circ}=\frac{nX}{X^2+Y^2+Z^2},\quad \eta-\eta^{\circ}=\frac{nY}{X^2+Y^2+Z^2},\quad \zeta-\zeta^{\circ}=\frac{nZ}{X^2+Y^2+Z^2},$$

qui résultent de notre analyse en faisant, pour abréger,

$$\frac{m}{A^2+B^2+C^2}=n, \text{ d'où } p^2=\frac{X^2+Y^2+Z^2}{n},$$

entraîneront l'équation demandée (1) dont la solution générale est exprimée ainsi d'une manière nouvelle et plus simple. En effet, on trouve d'abord

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{n^2[(X'-X)^2+(Y'-Y)^2+(Z'-Z)^2]}{(X^2+Y^2+Z^2)(X'^2+Y'^2+Z'^2)},$$

puis

$$(X'-X)^2+(Y'-Y)^2+(Z'-Z)^2=(x'-x)^2+(y'-y)^2+(z'-z)^2,$$

à cause des équations de condition entre α, β, etc. Et de là on tire

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{n^2[(x'-x)^2+(y'-y)^2+(z'-z)^2]}{(X^2+Y^2+Z^2)(X'^2+Y'^2+Z'^2)},$$

c'est-à-dire l'équation (1), en prenant

$$p^2=\frac{X^2+Y^2+Z^2}{n}=\frac{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}{n}.$$

223. On pourrait former inversement les valeurs de x, y, z en ξ, η, ζ; mais il est clair sans calcul, et à priori, que ces valeurs doivent s'exprimer par des formules du même genre que celles qui donnent ξ, η, ζ en x, y, z. En effet, p étant une fonction de x, y, z, on peut concevoir cette quantité comme fonction de ξ, η, ζ. Soit donc

$$p=\frac{1}{\varpi},\quad p'=\frac{1}{\varpi'},$$

ϖ étant une certaine fonction de ξ, η, ζ et ϖ' la même fonction de ξ', η', ζ'. L'équation (1) se changera dans l'équation nouvelle

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=\frac{(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2}{\varpi\varpi'},$$

d'une forme toute semblable à l'équation (1) elle-même, et qui, par conséquent, donnera x, y, z en ξ, η, ζ de la même manière que l'équation (1) a donné ξ, η, ζ en x, y, z.

224. On voit que, par l'échange des lettres x, y, z et ξ, η, ζ les unes dans les autres, une solution particulière de l'équation (1), je veux dire une solution dans laquelle les constantes auraient des valeurs particulières, en donnera une autre, la plupart du temps différente, quoique rentrant toujours, bien entendu, dans le type général indiqué tout à l'heure. Il est aisé aussi de voir que deux solutions données en fournissent une troisième. Supposons, en effet, qu'en prenant pour ξ, η, ζ q des fonctions de U, V, W, on ait

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{(U'-U)^2+(V'-V)^2+(W'-W)^2}{q^2q'^2},$$

et que, de même, en prenant pour U, V, W, p, des fonctions de x, y, z, on ait

$$(U'-U)^2+(V'-V)^2+(W'-W)^2=\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{p^2p'^2},$$

il est clair qu'on pourra exprimer aussi q, ξ, η, ζ en x, y, z, et qu'il viendra

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{p^2q^2.p'^2q'^2},$$

d'où une solution nouvelle de notre problème.

On peut dire, en d'autres termes, que diverses transformations qui résolvent ce problème étant opérées successivement, la transformation unique composée de cet ensemble le résout aussi. Et par la manière dont nous avons vérifié ci-dessus notre solution générale, il est manifeste que cette solution n'est que le résultat d'une suite de solutions particulières ainsi ajoutées entre elles pour ainsi dire.

225. Il y a une solution particulière de l'équation (1) que nous devons étudier spécialement parce qu'elle constitue, à proprement parler, l'élément essentiel de nos formules générales, et qu'elle nous servira d'ailleurs à en bien montrer le sens géométrique. Elle a été employée par M. Thomson, et consiste à poser

$$\xi=\frac{nx}{x^2+y^2+z^2},\quad \eta=\frac{ny}{x^2+y^2+z^2},\quad \zeta=\frac{nz}{x^2+y^2+z^2},$$

d'où résulte, en effet, l'équation

$$(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2=n^2\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{(x^2+y^2+z^2)(x'^2+y'^2+z'^2)},$$

c'est-à-dire l'équation (1), en prenant

$$p^2=\frac{x^2+y^2+z^2}{n}.$$

On a alors
$$\xi^2+\eta^2+\zeta^2=\frac{n^2}{x^2+y^2+z^2},$$
et, par conséquent,

$$x=\frac{n\xi}{\xi^2+\eta^2+\zeta^2},\quad y=\frac{n\eta}{\xi^2+\eta^2+\zeta^2},\quad z=\frac{n\zeta}{\xi^2+\eta^2+\zeta^2},$$

valeurs de même composition en ξ, η, ζ que les précédentes en x, y, z.

On peut interpréter géométriquement ces formules en regardant x, y, z, par exemple, comme des coordonnées rectangulaires, et ξ, η, ζ comme des paramètres. Les surfaces (ξ), (η), (ζ), pour lesquelles un de ces paramètres conserve même valeur, sont des sphères qui se coupent deux à deux orthogonalement, et par l'intersection de trois desquelles M. Thomson détermine la position de chaque point (x, y, z) ou (ξ, η, ζ). Sous ce point de vue, ξ, η, ζ sont des coordonnées curvilignes qui se rapportent à la même figure que les coordonnées rectilignes x, y, z. Mais il est plus commode, je crois, d'introduire dans nos recherches une de ces transmutations de figures si familières aux geomètres, et qui ont tant contribué aux progrès de la science dans ces derniers temps. La transformation dont il s'agit est bien connue, du reste, et des plus simples; c'est celle que M. Thomson lui-même a jadis employée sous le nom de principe des *images*.* Considérez x, y, z comme les coordonnées d'un point quelconque m d'une figure rapportée à trois axes rectangulaires Ox, Oy, Oz, ξ, η, ζ comme celles d'un point μ d'une autre figure rapportée à trois axes $O\xi$, $O\eta$, $O\zeta$ rectangulaires aussi, et auxquels nous donnons la même origine O, et respectivement les même directions, une de ces figures dérivant de l'autre, et le point μ, en particulier, correspondant au point m, en vertu des relations par lesquelles ξ, η, ζ s'expriment en x, y, z, ou x, y, z en ξ, η, ζ. Il est évident que les deux points correspondants m, μ sont en ligne droite avec l'origine O, et que le produit $Om.O\mu$ des rayons vecteurs Om, $O\mu$ est constant et $= n$. Une des figures se déduit donc de l'autre en prenant sur chacun des rayons vecteurs menés du point O à un point quelconque de la première figure d'autres rayons vecteurs en raison inverse des premiers; les extrémités de ces nouveaux rayons vecteurs déterminent la seconde figure. Nous donnerons à cette transformation le nom de transformation *par rayons vecteurs réciproques*, relativement à l'origine O. Si, pour un point m, on a $Om = \sqrt{n}$, on aura aussi $O\mu = \sqrt{n}$, et les points m et μ qui se correspondent ainsi dans les deux figures coïncideront. En disposant de n, on peut faire en sorte qu'un point donné m reste fixe dans la transformation; il suffit de prendre $n = \overline{Om}^2$, et alors tous les points situés sur la sphère dont O est le centre et Om le rayon, resteront fixes aussi, mais tous les autres seront déplacés.

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* Tome x. de ce Journal, page 364 [above, § 207].

226. A l'aide de cette transformation *par rayons vecteurs réciproques*, on déduira d'une figure donnée une infinité d'autres figures, soit en changeant l'origine O d'où partent les rayons vecteurs, soit en prenant diverses valeurs de n avec une même origine O, ce qui ne donne, au surplus, lieu qu'à des figures transformées toutes semblables entre elles, du moins tant que n garde le même signe; car les figures qui répondent à deux valeurs de n égales et de signes contraires sont symétriques. On peut d'ailleurs effectuer, l'une après l'autre, des transformations relatives à des origines différentes. Mais je dis que nos formules générales de n° 222 peuvent toujours s'interpréter à l'aide d'une seule transformation de cette espèce, en sorte qu'on n'obtiendrait rien de vraiment nouveau en ajoutant d'autres transformations à celle-là.

En effet, dans le cas le plus général, nous pouvons encore considérer x, y, z et ξ, η, ζ comme les coordonnées de deux points m, μ appartenant à deux figures différentes et rapportés à deux systèmes d'axes rectangulaires des x, y, z et ξ, η, ζ. Et voici comment s'opère la transformation de l'une des figures dans l'autre.

D'abord on passe de x, y, z, à X, Y, Z par les formules

$$X = \alpha(x - x_0) + \beta(y - y_0) + \gamma(z - z_0),$$
$$Y = \alpha'(x - x_0) + \beta'(y - y_0) + \gamma'(z - z_0),$$
$$Z = \alpha''(x - x_0) + \beta''(y - y_0) + \gamma''(z - z_0),$$

Or, à cause des équations de condition entre α, β, etc., ce passage n'est qu'un changement de coordonnées rectangulaires en d'autres coordonnées rectangulaires, qui n'altère en rien la première figure à laquelle il est appliqué ; on peut le supposer opéré d'avance, et confondre dès lors X, Y, Z avec x, y, z.

De là nous irons aux formules

$$\xi-\xi^0=\frac{nX}{X^2+Y^2+Z^2},\ \eta-\eta^0=\frac{nY}{X^2+Y^2+Z^2},\ \zeta-\zeta^0=\frac{nZ}{X^2+Y^2+Z^2},$$

et nous aurons ainsi une transformation de X, Y, Z en $\xi-\xi_0$, $\eta-\eta_0$, $\zeta-\zeta^0$, que nous regarderons comme des coordonnées rectangulaires prises par rapport aux mêmes axes. Cette transformation est à rayons vecteurs réciproques, comme nous l'avons vu n° 225. Elle s'opère en portant sur les rayons vecteurs menés de l'origine actuelle des longueurs inversement proportionnelles à ces rayons vecteurs ; l'ancienne figure se trouve ainsi changée en celle qui résulte des extrémités de toutes ces longueurs. Passer ensuite de $\xi-\xi^0$, $\eta-\eta^0$, $\zeta-\zeta^0$ à ξ, η, ζ n'est qu'un simple déplacement de l'origine, les axes restant parallèles à eux-mêmes ; cela ne produit dans la figure transformée aucune altération.

Nos formules du n° 222 résultent donc d'une transformation par rayons vecteurs réciproques, combinée avec des changements ordinaires de coordonnées. De telles transformations en nombre quelconque donnent toujours naissance à une équation de la forme (1), et l'interprétation géométrique des formules par lesquelles nous avions d'abord lié (n° 221) x, y, z et ξ, η, ζ semblait en demander deux, relatives à deux origines différentes, l'une pour le passage de X, Y, Z à u, v, w, l'autre pour le passage de u, v, w à ξ, η, ζ ; mais on voit, par ce qui précède, et grâce aux formules plus simples du n° 222, qu'une seule transformation suffit pour conduire au résultat le plus général ; il était important de le démontrer.

227. Les considérations géométriques dont nous venons de faire usage, pour interpréter les formules qui conduisent à l'équation (1), donnent lieu à des conséquences remarquables dont nous allons dire quelques mots. Dans les deux figures que déterminent respectivement les coordonnées x, y, z et les coordonnées ξ, η, ζ considérons, d'une part, deux points quel-

conques m, m', et, d'autre part, les points correspondants μ, μ'. Soient D la distance des deux premiers, Δ celle des deux autres, en sorte que

$$D^2=(x'-x)^2+(y'-y)^2+(z'-z)^2,$$

$$\Delta^2=(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2.$$

L'équation (1), qui pourra s'écrire

$$\Delta^2=\frac{D^2}{p^2p'^2},\quad \Delta=\frac{D}{pp'},\quad \frac{1}{\Delta}=\frac{pp'}{D},$$

fournit une relation entre la distance Δ de deux points μ, μ' dans l'une des figures et les quantités D, p, p'. Nous venons de dire que D est la distance des deux points m, m' correspondants dans l'autre figure; quant à p et p', ce sont, à un facteur constant près, les distances des points m, m' à un certain point fixe. Toute relation métrique entre deux ou plusieurs distances Δ dans l'une des figures fournira donc immédiatement une relation analogue dans l'autre figure. Mais il ne faut pas croire que les divers points correspondants à ceux de la droite Δ soient sur la droite D; cela arrive pour les points extrêmes par la définition même de ces droites, mais n'a pas lieu, en général, pour les points intermédiaires. En général, la suite des points correspondants à ceux d'une droite de la première figure forme dans la seconde figure une circonférence de cercle, laquelle ne se réduit à une ligne droite que dans un cas particulier, celui où son rayon est infini.

Ayant en ξ, η, ζ l'équation d'une surface ou les équations d'une ligne appartenant à la première figure, il suffit de substituer à ξ, η, ζ leurs valeurs pour former en x, y, z l'équation de la surface ou les équations de la ligne correspondante. On trouve bien facilement, de cette manière, que les plans et des sphères se transforment en des sphères qui peuvent se réduire à des plans quand le rayon devient infini; que, de même, des droites et des circonférences de cercle se transforment en des circonférences de cercle, etc. Mais, pour suivre le mécanisme de ces transformations, il suffit de considérer la transformation par rayons vecteurs réciproques, qui combinée avec des changements de coordonnées donne, comme on l'a vu, la transformation la plus générale. Soit donc

$$\xi=\frac{nx}{x^2+y^2+z^2}=\frac{nx}{r^2},\quad x=\frac{n\xi}{\xi^2+\eta^2+\zeta^2}=\frac{n\xi}{\rho^2},$$

$$\eta=\frac{ny}{x^2+y^2+z^2}=\frac{ny}{r^2},\quad y=\frac{n\eta}{\xi^2+\eta^2+\zeta^2}=\frac{n\eta}{\rho^2},$$

$$\zeta=\frac{nz}{x^2+y^2+z^2}=\frac{nz}{r^2},\quad z=\frac{n\zeta}{\xi^2+\eta^2+\zeta^2}=\frac{n\zeta}{\rho^2},$$

$$\Delta=\frac{D}{pp'}=\frac{nD}{\sqrt{(x^2+y^2+z^2)(x'^2+y'^2+z'^2)}}=\frac{nD}{rr'},$$

l'ensemble des formules relatives à la transformation par rayons vecteurs réciproques. On en conclut immédiatement ce que nous venons d'avancer, concernant les plans et les sphères, les droites et les circonférences de cercle. Mais on voit, de plus, et même sans calcul, que les plans qui passent par le point O, origine des rayons vecteurs, sont les seuls qui restent des plans dans la transformation; avant et après, leur position est la même, quoique leurs divers points, bien entendu, se soient déplacés pour se substituer les uns aux autres, ceux qui étaient loin de l'origine en étant à présent devenus voisins, et *vice versâ*. Tout autre plan se transforme en une sphère passant par le point O (où la transformation amène tous les points situés à l'infini) et ayant son centre sur la perpendiculaire au plan menée du point O; la perpendiculaire et le diamètre de la sphère ont un produit égal à la constante n, et se déduisent ainsi facilement l'une de l'autre. Il est inutile d'ajouter que deux sphères qui correspondent à deux plans parallèles se touchent au point O. De même, deux sphères ainsi posées se transformeraient en deux plans parallèles. Mais une sphère qui ne passe pas par le point O doit rester une sphère, puisqu'elle ne peut acquérir aucun point à l'infini. Les droites passant par le point O restent des droites, et conservent leur position invariable. Toute autre droite donne lieu à une circonférence de cercle dont le plan est déterminé par la droite et par le point O, et dont le centre est situé sur la perpendiculaire abaissée du point O sur la droite; le diamètre est le quotient de la constante n par cette perpendiculaire. Les circonférences provenant de droites parallèles sont toutes tangentes à une parallèle menée par le point O à ces droites. On peut voir, enfin, que la transformée d'une circonférence est une droite quand la circonférence passe par le point O, et, dans tout autre cas, reste une circonférence.

Une propriété remarquable de ce genre de transformation consiste en ce que les deux triangles formés par trois points infiniment voisins quelconques de la figure primitive et les trois points correspondants de sa transformée sont semblables l'un à l'autre, en sorte que si deux lignes se coupent dans l'une des deux figures sous un certain angle, les lignes correspondantes de l'autre figure se couperont sous le même angle.* La démonstration de cette propriété repose sur l'équation (1), à laquelle nous avons donné la forme

$$\Delta = \frac{D}{pp'}.$$

Supposons, en effet, que les deux points m, m', ou (x, y, z), (x', y', z'), soient infiniment voisins, et que leur distance D soit représentée par ds. Représentons par $d\sigma$ celle des deux points correspondants μ, μ'. Comme p et p' n'auront pas de différence sensible, il nous viendra

$$d\sigma = \frac{ds}{p^2}.$$

Les éléments $d\sigma$, ds ont donc en chaque lieu un rapport constant qui dépend de p et change, en général, d'un lieu à l'autre. Considérons un troisième point m'' infiniment voisin des deux premiers, et désignons par ds' et ds'' ses distances à m et à m'; $d\sigma'$, $d\sigma''$ étant les distances correspondantes dans la seconde figure, on aura encore

$$d\sigma' = \frac{ds'}{p^2},$$

$$d\sigma'' = \frac{ds''}{p^2}.$$

Donc $$d\sigma : d\sigma' : d\sigma'' :: ds : ds' : ds''.$$

Ainsi, le triangle infinitésimal $mm'm''$ est semblable au triangle correspondant $\mu\mu'\mu''$. L'angle de ds avec ds' est, par conséquent, le même que celui de $d\sigma$ avec $d\sigma'$. Cette démonstration, on le voit, n'exige pas même que l'équation (1) ait lieu pour deux points situés à une distance finie; elle demande seulement que cette équation ait toujours lieu pour deux points infiniment voisins. On doit en dire autant d'un théorème que je vais établir, et qui n'est qu'un corollaire de la proposition précédente.

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* De la similitude des triangles infiniment petits correspondants, il résulte encore que la figure transformée est semblable à la figure primitive, ou à sa symétrique, dans ses éléments infiniment petits. En s'en tenant au premier cas, qui est proprement celui de nos formules, où nous prenons naturellement la constante s positive, on aura, à trois dimensions, une sorte de représentations des corps, analogue au tracé des cartes géographiques [those according to the "stereographic projection"], pour lesquelles le rapport de similitude des éléments correspondants est variable aussi d'un lieu à l'autre.

Une surface appartenant à l'une des deux figures étant donnée, représentez-vous les lignes de courbure de cette surface, et les deux séries de surfaces développables, orthogonales entre elles et à la surface donnée, qui sont formées par les normales successives. Dans la seconde figure, les séries de surfaces correspondantes resteront orthogonales entre elles et à la transformée de la surface donnée; par suite, en vertu du beau théorème de M. Ch. Dupin, elles traceront encore sur cette transformée des lignes de courbure. Ces lignes de courbure résulteront ainsi des lignes de courbure de la première surface donnée, et seront immédiatement connues si les autres le sont. Il sera aisé d'appliquer ce théorème aux surfaces du second degré, comme aussi aux systèmes triples de surfaces orthogonales que M. Serret a indiqués dans une Note récente,* et qui, par notre transformation, en donneront d'autres non moins curieux, etc.

Proposons-nous, par exemple, de trouver les lignes de courbure de la surface enveloppe des sphères qui touchent trois sphères données, problème que M. Ch. Dupin a résolu jadis dans la *Correspondance sur l'École Polytechnique*, tome I, page 22. Soient O et P les points d'intersection de ces trois sphères; prenons le point O pour origine, et opérons une transformation par rayons vecteurs réciproques, ce qui nous fournira une seconde figure d'où nous reviendrons aisément à la première. Dans la seconde figure, les trois sphères données seront remplacées par trois plans qui se couperont en un point Π correspondant au second point P d'intersection de nos trois sphères. La surface enveloppe des sphères tangentes à ces trois plans sera (en se bornant à un des angles solides et à son opposé) celle d'un cône droit à base circulaire ayant son sommet au point *Π*, et circonscrit à une quelconque des sphères tangentes aux trois plans. Les lignes de courbure de cette surface conique sont: 1° les génératrices rectilignes qui passent toutes par le point *Π*: dans le retour à la première figure, ces droites deviendront des cercles passant tous par le point *P*, dont les tangentes en *P* feront toutes le même angle avec la tangente au cercle dans lequel se transforme l'axe du cône, d'où résultera un nouveau cône droit, et passant toutes aussi avec des circonstances semblables par le point *O*; 2° des cercles, dont les plans sont tous parallèles entre eux et perpendiculaires à l'axe du cône, et qui, lors du retour à la première figure, deviendront des cercles coupant à angle droit ceux qui résultent de génératrices rectilignes. Les lignes de courbure de la surface enveloppe des sphères tangentes à trois sphères données sont donc des circonférences de cercle.

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* Page 241 du présent volume [Liouville's *Journal*, 1847].

On démontre avec la même facilité le théorème de M. Dupin concernant la courbe que trace sur chacune des trois sphères données la sphère variable qui les touche. En effet, quand les trois sphères données sont remplacées par trois plans, il est clair que la suite des points suivant lesquels la sphère variable touche un quelconque des plans est une ligne droite passant par le point d'intersection *Π*. Donc, en revenant aux trois sphères données, la courbe demandée est une circonférence de cercle qui passe par les points *O* et *P*. Il peut arriver, bien entendu, que les points *O* et *P* soient imaginaires; mais il n'y a alors aucun changement essentiel à faire dans ce que nous venons de dire, et nos conclusions subsistent.

La circonstance d'une origine *O* imaginaire aurait plus d'inconvénient s'il s'agissait de résoudre le problème d'une sphère tangente à quatre autres, en le ramenant au problème très-simple de trouver une sphère tangente à une sphère donnée et à trois plans donnés; mais on y remédierait en augmentant d'une même quantité les rayons des quatre sphères données, ce qui ne change pas la position du centre de la sphère tangente. De même, en se bornant à considérer des points tous situés dans un plan passant par l'origine *O*, on ramènera la détermination du cercle tangent à trois autres à celle d'un cercle qui touche un cercle donné et deux droites données.

En général, les systèmes de sphères ou de cercles, et spécialement de sphères ou de cercles passant par un point donné, jouissent de propriétés curieuses dont beaucoup deviennent intuitives par la transformation dont nous venons de nous occuper. On peut appliquer en particulier cette remarque aux théorèmes que M. Miquel a donnés dans son Mémoire sur les angles curvilignes.* Pour nous borner au cas le plus simple, il est évident que, dans un triangle ABC formé par trois arcs de cercles passant tous par un même point O, la somme des angles vaut 2 droits, puisque notre transformation rend ce triangle rectiligne sans altérer ses angles.

228. Le passage des relations métriques d'une figure à l'autre, dans la transformation par rayons vecteurs réciproques, en allant des coordonnées ξ, η, ζ aux coordonnées x, y, z, s'opère à l'aide de la formule

$$\Delta = \frac{nD}{rr'},$$

ou simplement

$$\Delta = \frac{D}{rr'},$$

en posant $n=1$, ce qui n'a aucun inconvénient. Mais en désignant par O l'origine, dans la seconde figure seulement, et en employant les autres lettres A, B, etc.. pour représenter à la fois les points de la première figure et les points correspondants de la seconde figure, cette formule revient à dire que, dans toute relation entre des distances AB, BD, etc., il faut remplacer chaque distance telle que AB par $\frac{AB}{OA.OB}$. Voilà donc une règle pratique très-commode ; cette règle convient aussi bien au cas du plan qu'à celui de l'espace. Deux exemples suffiront.

Que des droites partant d'un point fixe A coupent chacune un cercle en deux points B et C, B' et C', etc., on aura

$$AB \times AC = AB' \times AC' = \text{constante}.$$

Donc, dans la figure transformée,

$$\frac{AB}{OA.OB} \times \frac{AC}{OA.OC} \times \frac{AB'}{OA.OB'} \times \frac{AC'}{OA.OC'},.$$

et par conséquent,

$$\frac{AB}{OB} \times \frac{AC}{OC} = \text{constante}.$$

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* Tome IX. de ce Journal, page 20 [Liouville's *Journal*, 1844].

D'ailleurs les points A, B, C, qui étaient en ligne droite, se trouvent à présent sur une circonférence de cercle passant par le point O. Nous voyons par là que les cercles passant par deux points fixes A, O coupent un cercle donné en deux points B, C tels, que le rapport des produits des distances $AB \times AC$ et $OB \times OC$ a une valeur constante pour tous ces cercles.

Que les côtés BC, AC, AB d'un triangle rectiligne ABC soient coupés en trois points A', B', C' par une transversale, on aura

$$AC' \times BA' \times CB' = BC' \times CA' \times AB'.$$

Donc, dans la figure transformée,

$$\frac{AC'}{OA.OC'} \times \frac{BA'}{OB.OA'} \times \frac{CB'}{OC.OB'} = \frac{BC'}{OB.OC'} \times \frac{CA'}{OC.OA'} \times \frac{AB'}{OA.OB'},$$

ce qui redonne

$$AC' \times BA' \times CB' = BC' \times CA' \times AB'$$

Mais cette relation s'applique à présent à un triangle curviligne ABC formé par trois cercles qui passent tous au point O et dont les côtés sont coupés en A', B', C' par un quatrième cercle passant aussi au point O. Il est, du reste, inutile d'ajouter que AC', BA', etc., sont les plus courtes distances des points A et C', B et A', etc., et non des segments mesurés sur les côtés du triangle curviligne.

On généraliserait aisément de la même manière le théorème relatif à un polygone gauche coupé par un plan. Mais en voilà assez sur ce sujet.

229. Etant données deux sphères qui ne se coupent pas, on peut toujours placer l'origine O sur la droite qui joint leurs centres, en un point réel tel, qu'après la transformation par rayons vecteurs réciproques, ces deux sphères seront concentriques. Prenons la droite des centres pour axe des x; designons par h la distance inconnue du point O au centre de la première sphère, et par $h+l$ sa distance au centre de la seconde sphère; soient k, k' les rayons. Les équations des deux sphères seront, avant la transformation,

$$(x-h)^2+y^2+z^2=k^2,$$
$$(x-h-l)^2+y^2+z^2=k'^2,$$

et après la transformation, qui consistera à remplacer x, y, z par

$$\frac{x}{x^2+y^2+z^2},\ \frac{y}{x^2+y^2+z^2},\ \frac{z}{x^2+y^2+z^2},$$

elles deviendront

$$\left(x-\frac{h}{h^2-k^2}\right)^2+y^2+z^2=\frac{k^2}{(h^2-k^2)^2},$$

$$\left[x-\frac{h+l}{(h+l)^2-k'^2}\right]^2+y^2+z^2=\frac{k'^2}{[(h+l)^2-k'^2]^2}.$$

Pour que le centre soit le même à présent, il faut et il suffit que

$$\frac{h}{h^2-k^2}=\frac{h+l}{(h+l)^2-k'^2},$$

d'où

$$lh^2+(l^2+k^2-k'^2)h+lk^2=0,$$

équation du second degré qui donnera pour h deux valeurs,

$$h=-\frac{l^2+k^2-k'^2}{2l}\pm\frac{1}{2l}\sqrt{G},$$

en posant

$$G=(l-k-k')(l-k+k')(l+k-k')(l+k+k');$$

et il est aisé de voir que G sera positive si les deux sphères qu'on a données d'abord ne se coupent pas.

230. Ce théorème pourra être utile en géométrie; mais il aura surtout une application importante dans les questions de physique mathématique. Essayons ici d'indiquer rapidement l'usage, en ce genre de questions, de la transformation générale qui donne l'équation (1). La Lettre de M. Thomson nous servira de guide; nous y ajouterons quelques développements. La généralité plus ou moins grande de la solution par laquelle on satisfait à l'équation (1) ne change en rien la marche à suivre, qui reste la même dans tous les cas.

Et d'abord de l'équation

$$\frac{1}{\Delta}=\frac{pp'}{D}$$

on peut conclure, avec M. Thomson, que, si une fonction U de ξ, η, ζ satisfait à l'équation

$$\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{d^2U}{d\zeta^2}=0,$$

cette même fonction, divisée par p et exprimée en x, y, z, vérifiera l'équation de même forme

$$\frac{d^2.p^{-1}U}{dx^2}+\frac{d^2.p^{-1}U}{dy^2}+\frac{d^2.p^{-1}U}{dz^2}=0.$$

De là une liaison entre deux problèmes distincts concernant tous deux l'équilibre de température dans les corps homogènes, mais relatifs à deux systèmes dont l'un résulte de l'autre par la transformation qui lie ξ, η, ζ à x, y, z.

Que le premier système soit formé de deux sphères qui ne se coupent pas, que la température soit donnée en chaque point de leurs surfaces, et demandons quelle est la loi des températures permanentes dans l'espace compris entre elles, si l'une est intérieure à l'autre, ou dans l'espace infini extérieur à toutes deux, si l'une est en dehors de l'autre, en ajoutant dans ce dernier cas la condition que la température soit nulle à l'infini. On ramènera cette question au cas très-facile de deux sphères concentriques. Cela résulte du théorème établi ci-dessus et en montre toute l'importance. En indiquant cette application à la théorie de la chaleur, M. Thomson ajoute, du reste, avec raison qu'elle s'étend d'elle-même à la théorie de l'électricité.

Dans la théorie de l'électricité ou du magnétisme, et, en général, dans la théorie de l'attraction, la quantité que G. Green et M. Gauss nomment *potentiel*, c'est-à-dire la quantité qu'on obtient en faisant la somme des éléments attractifs ou répulsifs d'une masse divisés par leurs distances à un point, joue un rôle capital. On connaît le problème de M. Gauss: "Distribuer sur une surface donnée une masse attractive ou répulsive, de telle sorte que le potentiel ait en chaque point de la surface une valeur donnée." On a résolu ce problème pour différentes surfaces, en particulier pour l'ellipsoïde. Or la solution relative à une surface quelconque donne la solution pour toutes les surfaces qui se déduisent de celle-là par une transformation pour laquelle l'équation (1) ait lieu. Ayant, en effet, l'équation

$$\iint \frac{\lambda' d\omega'}{\Delta} = Q$$

pour la première surface, on aura pour la seconde surface une équation du même genre, remplaçant par leurs nouvelles valeurs Δ et $d\omega'$. On a

$$\Delta = \frac{D}{pp'}.$$

Quant à $d\omega'$, j'observe que les éléments linéaires correspondants $d\sigma$ et ds sont liés par la formule

$$d\sigma = \frac{ds}{p^2}.$$

Donc entre deux éléments superficiels correspondants $d\omega$, ds,

on aura $$d\omega = \frac{ds}{p^4}, \quad d\omega' = \frac{ds'}{p'^4};$$

par suite, $$\iint \frac{\lambda'}{p'^2} \frac{ds'}{D} = \frac{Q}{p},$$

ce qui résout le problème de M. Gauss pour la surface transformée.

On peut voir aussi que les équations désignées par (*A*), (*B*), (*C*) dans mes Lettres à M. Blanchet,[*] et qui sont d'un si grand usage dans la plupart des questions physico-mathématiques concernant l'ellipsoïde, ont leurs analogues, qu'on en déduit immédiatement pour les surfaces transformées de l'ellipsoïde.[†]

On peut considérer encore l'équation

$$\frac{d^2U}{dt^2} = \frac{d^2U}{d\xi^2} + \frac{d^2U}{d\eta^2} + \frac{d^2U}{d\zeta^2},$$

et lui faire subir la transformation de ξ, η, ζ en x, y, z.

A cause de l'équation

$$d\sigma = \frac{ds}{p^2},$$

qui peut s'écrire

$$d\xi^2 + d\eta^2 + p\zeta^2 = \frac{dx^2 + dy^2 + dz^2}{p^4},$$

on trouve, par des formules connues, que la quantité

$$\frac{d^2U}{d\xi^2} + \frac{d^2U}{d\eta^2} + \frac{d^2U}{d\zeta^2}$$

est égale à

$$p^6 \left(\frac{d.\frac{1}{p^2}\frac{dU}{dx}}{dx} + \frac{d.\frac{1}{p^2}\frac{dU}{dy}}{dy} + \frac{d.\frac{1}{p^2}\frac{dU}{dz}}{dz} \right),$$

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[*] *Voyez* le tome XI de ce Journal.

[†] Parmi ces surfaces, il faut distinguer celle que donne la transformation *par rayons vecteurs réciproques*, en mettant l'origine au centre même de l'ellipsoïde. On sait qu'elle est aussi le lieu des pieds des perpendiculaires abaissées du centre sur les plans tangents à un autre ellipsoïde dont les axes ont pour valeurs les inverses des valeurs des axes de l'ellipsoïde donné. Une propriété analogue a lieu dans le plan, pour la lemniscate par exemple, qui peut ainsi être engendrée de deux manières différentes au moyen d'une hyperbole équilatère, circonstance dont M. Chasles a tiré un heureux parti dans ses recherches *sur les arcs égaux de la lemniscate* (*Comptes Rendus de l'Académie des Sciences*, tome XXI, séance du 21 juillet 1845.)

c'est-à-dire à

$$p^4\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right)-2p^3\left(\frac{dU}{dx}\frac{dp}{dx}+\frac{dU}{dy}\frac{dp}{dy}+\frac{dU}{dz}\frac{dp}{dz}\right)$$

ou enfin à $$p^5\left(\frac{d^2.p^{-1}U}{dx^2}+\frac{d^2.p^{-1}U}{dy^2}+\frac{d^2.p^{-1}U}{dz^2}\right),$$

en se rappelant que

$$\frac{d^2\frac{1}{p}}{dx^2}+\frac{d^2\frac{1}{p}}{dy^2}+\frac{d^2\frac{1}{p}}{dz^2}=0.$$

Par là on voit d'abord que l'équation

$$\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{d^2U}{d\zeta^2}=0$$

revient à celle-ci :

$$\frac{d^2.p^{-1}U}{dx^2}+\frac{d^2.p^{-1}U}{dy^2}+\frac{d^2.p^{-1}U}{dz^2}=0,$$

ce que nous savions déjà. On voit ensuite que l'équation

$$\frac{d^2U}{dt^2}=\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{d^2U}{d\zeta^2}$$

se transforme en

$$\frac{d^2U}{dt^2}=p^5\left(\frac{d^2.p^{-1}U}{dx^2}+\frac{d^2.p^{-1}U}{dy^2}+\frac{d^2.p^{-1}U}{dz^2}\right),$$

ou, mieux encore, en

$$\frac{d^2.p^{-1}U}{dt^2}=p^4\left(\frac{d^2.p^{-1}U}{dx^2}+\frac{d^2.p^{-1}U}{dy^2}+\frac{d^2.p^{-1}U}{dz^2}\right).$$

Réciproquement, cette dernière équation, où le coefficient p varie proportionnellement à la distance du point (x, y, z) à un point fixe, se ramène à l'équation

$$\frac{d^2U}{dt^2}=\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{d^2U}{d\zeta^2},$$

qui est à coefficients constants, résultat qui trouve une application utile dans la théorie du son.

On peut enfin ajouter que les équations aux différences partielles

$$\left(\frac{dU}{d\xi}\right)^2+\left(\frac{dU}{d\eta}\right)^2+\left(\frac{dU}{d\zeta}\right)^2=Q$$

et

$$\left(\frac{dU}{dx}\right)^2+\left(\frac{dU}{dy}\right)^2+\left(\frac{dU}{dz}\right)^2=\frac{Q}{p^4}$$

sont des transformées l'un de l'autre, ce qui pourra servir dans les questions de dynamique, où MM. Hamilton et Jacobi ont introduit de telles équations aux différences partielles.

On me pardonnera, je l'espère, ces développements que j'ai cru pouvoir donner, à la suite des deux Lettres si intéressantes de M. Thomson, sans le gêner dans ses recherches. Mon but sera rempli, je le répète, s'ils peuvent aider à bien faire comprendre la haute importance du travail de ce jeune géomètre, et si M. Thomson lui-même veut bien y voir une preuve nouvelle de l'amitié que je lui porte et de l'estime que j'ai pour son talent.

XV. DETERMINATION OF THE DISTRIBUTION OF ELECTRICITY ON A CIRCULAR SEGMENT OF PLANE OR SPHERICAL CONDUCTING SURFACE, UNDER ANY GIVEN INFLUENCE.

[Jan. 1869. *Not hitherto published.*]

231. The electric density at any point of the surface of an insulated conducting ellipsoid, electrified and left undisturbed by external influence, is (§ 11) simply proportional to the distance of the tangent plane from the centre. If we take $\rho = kp$ as the expression of this law, and call q the whole quantity of electricity communicated, we have (§ 14) $4\pi kabc = q$; so that the formula for the electric density, ρ, at any point P of the surface in terms of p, the distance of the tangent plane from the centre, and a, b, c the three semi-axes, is

$$\rho = \frac{q}{4\pi abc} p \qquad (1);$$

or, in terms of rectangular co-ordinates of the point P,

$$\rho = \frac{q}{4\pi abc\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}} \qquad (2).$$

232. To find the "electrostatic capacity" (§ 51, footnote) of the charged ellipsoid, let V denote the potential at its surface.

We have, by § 15 (e),

$$V = q\int_a^\infty \frac{du}{\sqrt{(u^2 - a^2 + b^2)}\sqrt{(u^2 - a^2 + c^2)}} \qquad (3);$$

and therefore the capacity is the reciprocal of the definite integral which appears in this formula.

233. By taking $c = 0$ we fall on the case of an infinitely thin plane elliptic disc: for which we have

$$c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) + \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2};$$

and therefore

$$\rho=\frac{q}{4\pi ab\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}} \qquad (4)$$

Putting $b=a$ in this, we have, for an infinitely thin circular disc,

$$\rho=\frac{q}{4\pi a(a^2-r^2)^{\frac{1}{2}}} \qquad (5)$$

where a denotes the radius of the disc, and ρ the electric density on either side of it, at a distance r from the centre. This result was first given by Green, near the conclusion of his paper "On the Laws of the Equilibrium of Fluids analogous to the Electric Fluid" (*Transactions of the Cambridge Philosophical Society* for Nov. 12, 1832); from which I make the following extract:—

234. "Biot (*Traité de Physique*, tome ii. p. 277) has related the results of some experiments made by Coulomb on the distribution of the electric fluid when in equilibrium upon a plate of copper 10 inches in diameter, but of which the thickness is not specified. If we conceive this thickness to be very small compared with the diameter of the plate, which was undoubtedly the case, the formula just found ought to be applicable to it, provided we except those parts of the plate which are in the immediate vicinity of its exterior edge. As the comparison of any results mathematically deduced from the received theory of electricity with those of the experiments of so accurate an observer as Coulomb must always be interesting, we will here give a table of the values of the density at different points on the surface of the plate, calculated by means of the formula (29), together with the corresponding values found from experiment:—

Distances from the Plate's edge.	Observed Densities.	Calculated Densities.
5 in. . . .	1,	1
4	1,001	1,020
3	1,005	1,090
2	1,17	1,250
1	1,52	1,667
,5. . . .	2,07	2,294
0	2,90	infinite

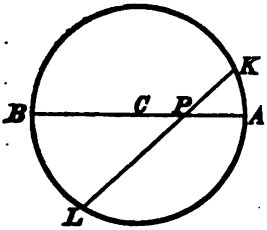
"We thus see that the differences between the calculated and observed densities are trifling; and, moreover, that the "observed are all something smaller than the calculated ones, "which, it is evident, ought to be the case, since the latter "have been determined by considering the thickness of the "plate as infinitely small, and consequently they will be some-"what greater than when this thickness is a finite quantity, as "it necessarily was in Coulomb's experiments."

235. In this case (3) of § 232 becomes

$$V=q\int_a^\infty \frac{du}{u\sqrt{(u^2-a^2)}}=q\frac{\pi}{2a} \qquad (6).$$

Hence the capacity is $\frac{2a}{\pi}\cdot$ But [§ 232 (3)] the capacity of a globe is numerically equal to its radius; and therefore the capacity of an infinitely thin disc is less than that of a globe of equal radius, in the ratio of 1 to $\frac{\pi}{2}$, or 1 to 1·571. Cavendish found the ratio 1 to 1·57, by experiment![*]

236. The expression (5), § 233, for the electric density at any point P on either side of an infinitely thin circular disc of conducting material electrified and left free from disturbing influence, may be put into a form more convenient for geometrical investigation, thus:—Let C be the centre of the disc, so that $CA=a$, $CP=r$, according to previous notation. Hence $BP=a+r$; $PA=a-r$;



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[*] My authority for this statement is the following entry which I find written in pencil on an old memorandum-book:—

"PLYMOUTH, *Mond.*, *July* 2, 1849.

"Sir William Snow Harris has been showing me Cavendish's unpublished "MSS., put in his hands by Lord Burlington, and his work upon them; a "most valuable mine of results. I find already the capacity of a disc "(circular) was determined experimentally by Cavendish as $\frac{1}{1{\cdot}57}$ that of a "sphere of same radius. Now we have

$$\text{"capacity of disc}=\frac{\int_0^a \frac{rdr}{(a^2-r^2)^{\frac{1}{2}}}}{\int_0^a \frac{dr}{(a^2-r^2)^{\frac{1}{2}}}}=\frac{a}{\frac{\pi}{2}}=\frac{a}{1{\cdot}571}\,!\text{"}$$

It is much to be desired that those manuscripts of Cavendish should be published complete; or, at all events, that their safe keeping and accessibility should be secured to the world.

$$a^2 - r^2 = BP.PA = KP.PL;$$

if KL be any chord through P; and (5), with $\frac{2aV}{\pi}$ substituted for q according to (6), becomes

$$\rho = \frac{V}{2\pi^2\sqrt{KP.PL}} \qquad (7).$$

237. Consider a plane disc, S', thus electrified, to a potential which we shall, for a moment, denote by V'; and, following the suggestion of § 210, take its image relatively to a spherical surface of radius R described from any point Q as centre. This image will (§ 207) be a spherical segment, S, electrified (§§ 210 and 238) as an infinitely thin conducting surface under the influence of a quantity $V'R$ of electricity concentrated at Q; and (compare § 213) the spherical surface of which S is a part will pass through Q. The reader will have no difficulty in verifying these statements for himself; but if he desires it, he will find some further information and examples in Thomson and Tait's *Natural Philosophy*, §§ 512...518. Thus (§ 515 of that work) if ρ' be the electric density on either side of the disc at P'_1, and ρ that on either side of its image at P, we have

$$\rho = \frac{-R^3}{QP^3}\rho' \qquad (8);$$

and if v' and v be the potentials at any point Π', and Π the image of Π', due respectively to the disc S' and its image, we have (Thomson and Tait, § 516)

$$v = \frac{R}{Q\Pi}v' \qquad (9).$$

This shows that, as the potential due to S' has a constant value, V', at all points of S', the potential due to S will be, at different points of S, inversely as their distances from Q; and if we take $q = -RV'$, and denote by V the potential due to electricity distributed over the two sides of S, we have

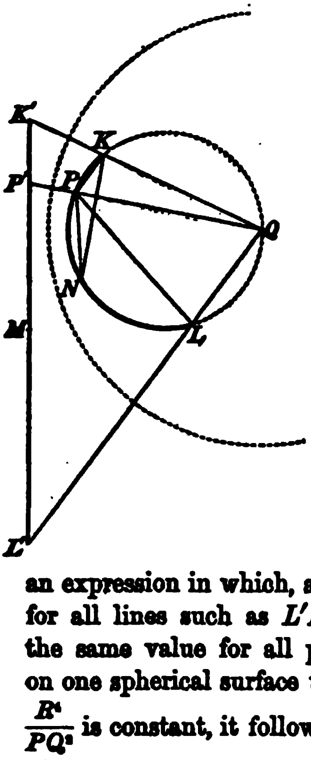
$$V = \frac{-q}{QP} \qquad (10),$$

and so see that S is electrified as an infinitely thin conducting sheet of the same figure would be if connected with the earth by an infinitely fine wire, and inductively electrified only by the influence of a quantity $-q$ of electricity insulated at Q. Now, with our present notation (7) gives

$$\rho' = \frac{V'}{2\pi^2\sqrt{L'P'.P'K'}} = -\frac{1}{2\pi^2\sqrt{L'P'.P'K'}}\cdot\frac{q}{R} \qquad (11),$$

if $K'L'$ be any chord through P' of the circle bounding the plane disc S'.

238. Let K, P, and L be the images of K', P', L', so that KL, the image of $K'L'$, is the arc in which S is cut by the plane through Q and $K'L'$. We have (§ 207)



$$K'Q = \frac{R^2}{KQ}, \quad P'Q = \frac{R^2}{PQ}.$$

Hence $K'Q : P'Q :: PQ : KQ$; and therefore the triangles $K'P'Q$, PKQ are similar; and therefore

$$P'K' = KP.\sqrt{\frac{P'Q.K'Q}{KQ.PQ}} = KP\frac{R^2}{KQ.PQ} \quad (12).$$

(Compare §§ 213, 227.) From this, and the corresponding expression for LP, we have

$$L'P'.P'K' = \frac{R^4}{PQ^2}\cdot\frac{LP.KP}{LQ.KQ} \quad (13);$$

an expression in which, as the first member has the same value for all lines such as $L'K'$ through P', the second must have the same value for all planes through PQ, cutting one circle on one spherical surface through P and Q, in K and L. As $\frac{R^4}{PQ^2}$ is constant, it follows that $\frac{LP.KP}{LQ.KQ}$ is constant;* a theorem of geometry given above (§ 228) by M. Liouville. Each member of (12) may be altered in form thus: bisect $L'K'$ in M, and arc LK in N. We have

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* As a particular case let Q be either pole of the fixed circle. In this case $LQ = KQ$, a constant. Hence $LP.KP$ is constant; that is, the product of the two chords from any fixed point P on a spherical surface to the two points in which any fixed circle on the surface is cut by a plane through P and one of the poles of that circle, is constant, however the plane be varied. This is the simplest extension to spherical surfaces of the elementary geometrical theorem (Euc. III. 35) for the constancy of the rectangle under the two parts of a varying chord of a fixed circle through a fixed point, already used in the text (§ 236).

$L'P' = MK'+$

$$\left.\begin{aligned} L'P'.P'K' &= MK'^2 - MP'^2 \\ LP.KP &= NK^2 - NP^2 \\ LQ.KQ &= NQ^2 - NK^2 \end{aligned}\right\} \qquad (14),$$

equations of which the last two are very easily proved from the formula $\sin(\alpha-\beta)\sin(\alpha+\beta) = \sin^2\alpha - \sin^2\beta$,

by taking for α and β the angles subtended by $\frac{1}{2}NK$ and $\frac{1}{2}NP$ at the centre of the circle $QKPL$; and again, by taking for α and β the angles subtended by $\frac{1}{2}NQ$ and $\frac{1}{2}NK$ at the same point.

239. Using (13) in (11), and the result in (8), we find

$$\rho = \frac{1}{2\pi^2}\frac{q}{QP^2}\sqrt{\frac{LQ.KQ}{LP.KP}} \qquad (15);$$

and modifying by (14),

$$\rho = \frac{1}{2\pi^2}\frac{q}{QP^2}\sqrt{\frac{NQ^2 - NK^2}{NK^2 - NP^2}} \qquad (16).$$

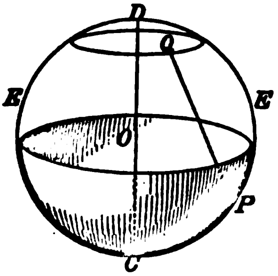
If we take for $PKQL$ the plane through PQ and C the central point (or pole) of the spherical segment S, so that N becomes C, NK becomes equal to the chord of any arc from C to the lip; and (15) becomes

$$\rho = \frac{1}{2\pi^2}\frac{q}{QP^2}\sqrt{\frac{CQ^2 - a^2}{a^2 - CP^2}} \qquad (17),$$

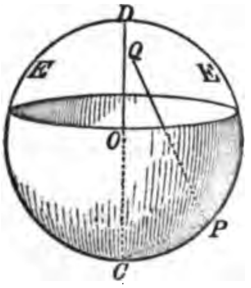
which is the result stated in § 219, above. It is remarkable that this expression is independent of the radius of the spherical surface of which the bowl is a part. Hence, if we suppose the radius infinite, we have the same expression (17) for the electric density at any point P on either side of an infinitely thin circular disc of radius a, connected with the earth by an infinitely fine wire, and influenced by a quantity Q of electricity collected at any point Q in the plane of the disc, but outside its bounding circle. It agrees with the solution previously given by Green for this case in his paper referred to in § 234, above.

240. (Compare § 220.) To find the distribution for the case in which S is insulated, electrified, and removed from all disturbing influence, let V be the constant potential produced throughout S by this distribution. Remark that the same

distribution of electricity on S would be produced inductively if it were connected by an infinitely fine wire with the earth, and enclosed by any surface, EE, rigidly electrified with such a quantity and distribution of electricity as (§§ 5, 73, 206, 207; also Thomson and Tait, § 499) to produce a uniform potential $-V$ through its interior. Now take this enclosing surface, EE, to be spherical, concentric with that of which S is a part, and of radius greater than that of the last mentioned by an infinitely small excess. The electric density of the inducing distribution will be uniform all over EE, and equal to $-\frac{V}{2\pi f}$ if f be the diameter of the surface. The portion of EE which lies infinitely near to the convex surface of S will clearly induce on this convex surface an equal electric density of contrary sign, that is, $+\frac{V}{2\pi f}$. The remainder of EE will induce equal electric densities on the concave and convex sides of S, the amount of either of which at any point P is to be obtained by integration from (15), (16), or (17), thus:—



241. Let $d\sigma$ be an infinitesimal element of E, situated at a point Q anywhere on it. The quantity of electricity on this element is $-\frac{Vd\sigma}{2\pi f}$; and using this for $-q$ in (17), we find, for the density on either side at P, of the electrification induced by it, the following expression:—



$$\frac{V}{4\pi^2 f}\frac{d\sigma}{PQ^2}\sqrt{\frac{CQ^2-a^2}{a^2-CP^2}}.$$

Now calling O the centre of the spherical surface, let COP be denoted by η; COQ by θ; the value of either of these when P or Q is at the lip of the bowl, by α; and the angle between the planes of COP and COQ, by ϕ: so that we have

$$a^2=\tfrac{1}{2}f^2(1-\cos\alpha),\quad CP^2=\tfrac{1}{2}f^2(1-\cos\eta),\quad CQ^2=\tfrac{1}{2}f^2(1-\cos\theta)$$

and $$PQ^2=\tfrac{1}{2}f^2(1-\cos\eta\cos\theta-\sin\eta\sin\theta\cos\phi);$$

and we may take

$$d\sigma = \tfrac{1}{2} f^2 \sin\theta \, d\theta \, d\phi.$$

Hence if, lastly, ρ denote the electric density at P, on the concave side of the segment, we have

$$\rho = \frac{V}{8\pi^3 f \sqrt{(\cos\eta - \cos\alpha)}} \int_\alpha^\pi d\theta \sin\theta \sqrt{(\cos\alpha - \cos\theta)} \int_0^{2\pi} \frac{d\phi}{1 - \cos\eta\cos\theta - \sin\eta\sin\theta\cos\phi}.$$

But putting $\tan\tfrac{1}{2}\phi = t$, $A = 1 - \cos\eta\cos\theta$, and $B = \sin\eta\sin\theta$, we find

$$\int_0^{2\pi} \frac{d\phi}{1 - \cos\eta\cos\theta - \sin\eta\sin\theta\cos\phi} = 4\int_0^\infty \frac{dt}{A - B + (A+B)t^2} = \frac{2\pi}{\sqrt{(A^2 - B^2)}} = \frac{2\pi}{\cos\eta - \cos\theta},$$

and therefore

$$\rho = \frac{V}{4\pi^2 f \sqrt{(\cos\eta - \cos\alpha)}} \int_\alpha^\pi \frac{d\theta \sin\theta \sqrt{(\cos\alpha - \cos\theta)}}{\cos\eta - \cos\theta}.$$

Lastly, putting $\sqrt{(\cos\alpha - \cos\theta)} = z$, we find

$$\int_\alpha^\pi \frac{d\theta \sin\theta \sqrt{(\cos\alpha - \cos\theta)}}{\cos\eta - \cos\theta} = 2\int_0^{\sqrt{(\cos\alpha + 1)}} \frac{z^2 dz}{\cos\eta - \cos\alpha + z^2}$$

$$= 2\left\{ \sqrt{(\cos\alpha + 1)} - \sqrt{\cos\eta - \cos\alpha}\,\tan^{-1}\sqrt{\frac{\cos\alpha + 1}{\cos\eta - \cos\alpha}} \right\}.$$

Hence we have, in conclusion,

$$\rho = \frac{V}{2\pi^2 f}\left\{ \sqrt{\frac{\cos\alpha + 1}{\cos\eta - \cos\alpha}} - \tan^{-1}\sqrt{\frac{\cos\alpha + 1}{\cos\eta - \cos\alpha}} \right\} \qquad (18),$$

or, with f and a as above, and r to denote the chord CP,

$$\rho = \frac{V}{2\pi^2 f}\left\{ \sqrt{\frac{f^2 - a^2}{a^2 - r^2}} - \tan^{-1}\sqrt{\frac{f^2 - a^2}{a^2 - r^2}} \right\} \qquad (19);$$

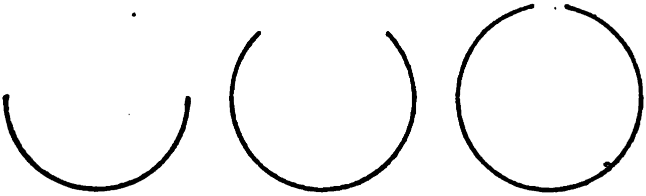
and the same, with the addition of

$$\frac{V}{2\pi f} \qquad (20),$$

gives (§ 240) the electric density on the convex side; which are exactly the results stated above in § 220. Twenty-two years ago these and the very simple formula (17) were communicated by me to M. Liouville without proof, and were published in his Journal. From that time till now they have not been proved, or even noticed, so far as I am aware, by any other writer.

242. Numerical results, calculated from the preceding formulæ (19) and (20), are shown in the following tables:—

Plane Disc.	Curved Disc. Arc 10°.			Curved Disc. Arc 20°.			Bason. Arc 90°.		
	Concave.	Convex.	Mean.	Concave.	Convex.	Mean.	Concave.	Convex.	Mean.
1·0000	·9136	1·0685	1·0000	·8636	1·1364	1·0000	·4459	1·5541	1·0000
1·0142	·9457	1·0826	1·0141	·8776	1·1504	1·0140	·4469	1·5551	1·0010
1·0607	·9920	1·1289	1·0605	·9236	1·1964	1·0600	·4828	1·5910	1·0369
1·1547	1·0858	1·2227	1·1542	1·0165	1·2893	1·1529	·5566	1·6648	1·1106
1·3416	1·2722	1·4091	1·3407	1·2884	1·5611	1·4247	·7065	1·8147	1·2606
1·8091	1·7386	1·8755	1·8071	1·6652	1·9379	1·8016	1·0933	2·2015	1·6474



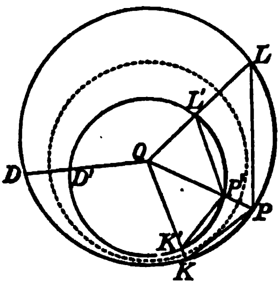
Bowl. Arc 180°.			Bowl. Arc 270°.			Bowl. Arc 340°.		
Concave.	Convex.	Mean.	Concave.	Convex.	Mean.	Concave.	Convex.	Mean.
·1202	1·8798	1·0000	·0135	1·9865	1·0000	·0001	1·9999	1·0000
·1266	1·8862	1·0064	·0144	1·9874	1·0009	·0002	1·9999	1·0000
·1418	1·9014	1·0216	·0176	1·9906	1·0041	·0002	2·0000	1·0001
·1779	1·9375	1·0577	·0253	1·9983	1·0118	·0004	2·0001	1·0002
·2570	2·0166	1·1366	·0451	2·0181	1·0316	·0009	2·0006	1·0007
·4959	2·2555	1·3757	·1195	2·0925	1·1060	·0042	2·0040	1·0041

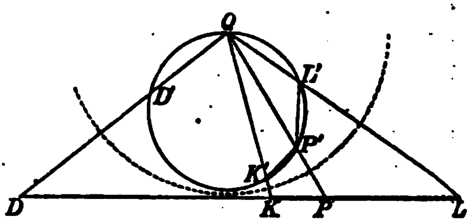
It is remarkable how slight an amount of curvature produces a very sensible excess of electric density on the convex side in the first two cases (10° and 20°) of curved discs; yet how nearly the mean of the densities on the convex and concave sides at any point agrees with that at the corresponding point on a plane disc shown in the first column. The results for bowls of 270° and 340° illustrate the tendency of the whole charge to the convex surface, as the case of a thin spherical conducting surface with an infinitely small aperture is approached.

The constant coefficient for each case has been taken so as to make the mean of the electric densities on the convex and concave sides unity at the middle point (as in Green's numbers, § 234 above, for the plane disc). The six points for which the electric densities are shown in the tables below are (not the six points to which Coulomb's observations and Green's numbers quoted in § 234 refer, but) the middle point, and the five points dividing the arc from the middle to the edge or lip into six equal parts.

243. A second application of the principle stated in § 210, and used in §§ 237...239, allows us to proceed from the solution now found for the electrification of an uninfluenced bowl to determine the electrification of a bowl or disc under the influence of electricity insulated at a point Q (not, as in the solution of § 239, necessarily in the spherical surface or plane of the bowl or disc, but) *anywhere in the neighbourhood.* Consider the image, S, of an uninfluenced electrified bowl, S', relatively to a spherical surface described from any point Q in its neighbourhood, as centre, with radius R. Let D' be the point on the spherical surface of S' continued, which is equidistant from the lip (so that D' and the middle point of the conducting surface S' are the two poles of the circle constituting the lip); $D'K'P'L'$ the circle in which S', and the continuation of its spherical surface, are cut by the plane through D', Q, and any point P of S at which it is desired to find the electric density; and $DKPL$ the image of $D'K'P'L'$.

In the annexed diagrams two cases are illustrated; in one of which S is spherical and concave towards the influencing point, Q; in the other, S is plane. Using now for S' all the notation of §§ 240, 241, but with accents added, and taking advantage of § 238, footnote, we see that





$$a'^2 - r'^2 = K'P'.P'L'$$

and $$f'^2 - a'^2 = D'K'^2 = D'L'^2 = D'K'.D'L'.$$

Hence (19) becomes

$$\rho' = \frac{V'}{2\pi^2 f'}\left\{\sqrt{\frac{D'K'.D'L'}{P'K'.P'L'}} - \tan^{-1}\sqrt{\frac{D'K'.D'L'}{P'K'.P'L'}}\right\} \qquad (21),$$

for the electric density at P' on the concave side of S'. But, as in § 238, we find

$$P'K' = PK.\frac{R^2}{QP.QK}, \quad P'L' = PL.\frac{R^2}{QP.QL} \qquad (22),$$

and $$D'K' = D'L' = DK\frac{R^2}{QD.QK} = DL\frac{R^2}{QD.QL} \qquad (23).$$

Also, if h denote the shortest distance from Q to the spherical or plane surface of S, and f the diameter of this surface (infinite of course when the surface is plane, or negative if the convexity be towards Q), we have

$$f' = \frac{R^2}{h} + \frac{R^2}{f-h} = \frac{R^2 f}{h(f-h)} \qquad (24).$$

Using these in (21), putting $V' = \frac{q}{R}$, and substituting the expression so obtained for ρ' in (8) of § 237, we find

$$\rho = \frac{qh(f-h)}{2\pi^2 f.PQ^2}\left\{\frac{PQ}{DQ}.\sqrt{\frac{DK.DL}{PK.PL}} - \tan^{-1}\left[\frac{PQ}{DQ}\sqrt{\frac{DK.DL}{PK.PL}}\right]\right\} \qquad (25),$$

for the electric density on the side of S *remote* from Q (that is, the convex or concave side, when S is spherical, according as Q is within or without the completed spherical surface). The electric density on the side next Q is [§ 241 (20)] the same, with the addition of

$$\frac{qh(f-h)}{2\pi f.PQ^2} \qquad (26).$$

These formulæ, (25) and (26), express the electric density on the two sides of a circular segment or disc of infinitely thin spherical or plane conducting surface connected with the earth by an infinitely fine wire, and electrified by the influence of a quantity $-q$ of electricity insulated at a point Q anywhere in its neighbourhood.

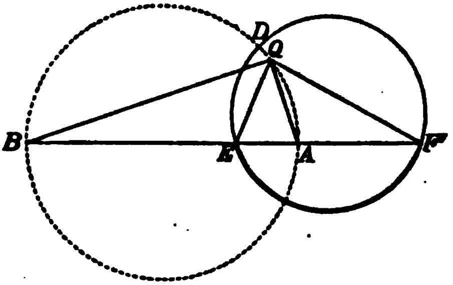
244. The position of the auxiliary point D (which appears in the diagrams as the image of D', the unoccupied pole of the lip of the original bowl S') may be found, without reference to S', by construction from S and Q supposed given; thus:—From (22) of § 243 we have

$$KD : DL :: KQ : QL \qquad (27),$$

where K and L may be the points in which the lip of the bowl S is cut by *any* plane through $QD'D$. Let, for instance, this plane pass through the centre of one of the spherical surfaces. It must also pass through the centre of the other, and bisect each bowl; and if E, F be the points in which it cuts the lip of S, (26) applied to the present case gives

$$ED : DF :: EQ : QF.$$

Hence (Euclid, VI. 3) the lines bisecting the angles EDF, EQF cut the base EF in the same point; and D must be in the circle which is the locus of all points in the plane EFQ fulfilling this condition, being found by the well-known construction, thus:—Bisect the angle EQC by QA, meeting EF in A. Draw QB perpendicular to QA, and let it meet EF produced, in B. On BA as diameter describe a circle, which is the required locus; and D is the point in which this circle cuts the unoccupied part of the spherical or plane surface of S.



245. D being found by this simple construction, the solution of the problem is complete, without reference to S', thus:—To find the electric density at any point P, draw a plane through QDP, and let it meet the lip in K and L. Measure DK, DL, PK, PL, PQ, and DQ, and calculate by (25) and (26). But we have an important simplification from the geometrical theorem of § 238, which shows that

$$\frac{DK.DL}{PK.PL}=\frac{Dk.Dl}{Pk.Pl} \qquad (28)$$

if k, l be points in which the lip is cut by any plane whatever through PD. Choose, for instance, the plane through PD, and C the middle point of S. Then, as D, k, P, C, l lie all on one circle, and C is the middle point of the arc kPl, we have (as above, in § 238)

$$Dk.Dl = CD^2 - Ck^2 = CD^2 - a^2,$$
$$Pk.Pl = Ck^2 - CP^2 = a^2 - CP^2;$$

where, as before, a denotes the chord from the middle point to the lip. Using this in (28) and (25) we have, finally,

$$\rho=\frac{qh(f-h)}{2\pi^2 fPQ^3}\left\{\frac{PQ}{DQ}\sqrt{\frac{CD^2-a^2}{a^2-CP^2}}-\tan^{-1}\left[\frac{PQ}{DQ}\sqrt{\frac{CD^2-a^2}{a^2-CP^2}}\right]\right\} \qquad (29),$$

for the density on the side remote from Q; h and $f-h$ being the shorter and longer distance.

246. For the case in which S is a plane disc, or $f=\infty$, this becomes

$$\rho=\frac{qh}{2\pi^2 PQ^3}\left\{\frac{PQ}{DQ}\sqrt{\frac{CD^2-a^2}{a^2-CP^2}}-\tan^{-1}\left[\frac{PQ}{DQ}\sqrt{\frac{CD^2-a^2}{a^2-CP^2}}\right]\right\} \qquad (30),$$

and the addition (26) to it to give the electric density on the side next to Q,

$$\frac{qh}{2\pi PQ^3} \qquad (31).$$

Also, as EFD is a straight line in this case, (27) gives

$$CD=\tfrac{1}{2}EF\frac{QF+QE}{QF-QE} \qquad (32).$$

QE, QF are to be calculated immediately from data of whatever form, specifying the position of Q; and from them and CD found by this formula, DQ is to be calculated. Thus explicitly we have every element required for calculating electric densities by (30).

247. For the case of Q in the axis of the disc, D is infinitely distant, so that $CD=\infty$, $DQ=\infty$, and $\frac{CD}{DQ}=1$. And if for CP we put r, (30) and (31) give, for the density on remote side,

$$\rho=\frac{qh}{2\pi^2 PQ^3}\left\{\frac{PQ}{(a^2-r^2)^{\frac{1}{2}}}-\tan^{-1}\frac{PQ}{(a^2-r^2)^{\frac{1}{2}}}\right\} \qquad (33),$$

and for the density on near side,

$$\rho+\frac{qh}{2\pi PQ^3} \qquad (34).$$

If P be at the centre of the disc, and if we take $q=2\pi^2$, these become

$$\left.\begin{aligned}&\text{for remote side, } \rho=\frac{1}{h^2}\left(\frac{h}{a}-\tan^{-1}\frac{h}{a}\right)\\ &\text{for near side, } \quad \rho+\frac{\pi}{h^2}\end{aligned}\right\} \qquad (35);$$

from which the following numerical results have been calculated, with a, the radius of the disc taken as unity:—

Distance of Influencing Point h.	Induced Electric Density at middle of Disc	
	On remote side $\rho = h^{-1} - \cot^{-1}(h^{-1})$.	On near side $\rho + \pi h^{-2}$.
·2	·1651	78·7049
·4	·1218	19·7567
·6	·1655	8·8921
·8	·1957	5·1044
1·0	·2146	3·3562
1·2	·2250	2·4067
1·4	·2293	1·8322
1·6	·2296	1·4568
1·8	·2273	1·1969
2·0	·2232	1·0086
3·0	·1946	·5437

248. These numbers show that the distance at which the influencing point, if restricted to the axis of the disc, must be held to render the induced electric density at the middle on the far side a maximum is about 1·5 times the radius. But the characteristics (1.) of the zero electric density on the far side, and infinite on the near side, when the influencing point is infinitely near the disc; (2.) the proportionality of the latter to h^{-2} for very small distances; and (3.) the ultimate vanishing of the difference between the two sides as the influencing point is removed to an infinite distance, and the approximation of each to Green's result for a plane uninfluenced disc electrified to a potential equal to qh (§ 234, above), is better illustrated by the formulæ themselves, (35); (33), (34); and (30), (31); than by any numerical results calculated from them, however elaborately. It would be interesting to continue the analytical investigation far enough to determine the electric potential at any point in the neighbourhood of a disc electrified under influence, and so to illustrate further than is done by the numbers and formulæ already obtained, the theory of *electric screens*, and of Faraday's celebrated "induction in curved lines" (*Experimental Researches in Electricity*, §§ 1161, 1232; Dec. 21, 1837); but I am obliged to leave the subject for the present, in the hope that others may be induced to take it up.

XVI. ATMOSPHERIC ELECTRICITY.

[From Nichol's *Cyclopædia*, 2d Ed. (1860).]

249. It may be premised, to avoid circumlocution in this article, that every body in communication with the earth by means of matter possessing electric conductivity enough to prevent its electric potential* from differing sensibly from that of the earth, will be called part of the earth. Moist stone, and rock of all kinds, and all vegetable and animal bodies, in their natural conditions, except in circumstances of extraordinary dryness, possess, either superficially or throughout their substance, the requisite conductivity to fulfil that condition. On the other hand, various natural minerals and artificial compounds, such as glass,—various vegetable gums, such as India-rubber, gutta percha, rosin,—and various animal products, such as silk and gossamer fibre,—when either in a very dry natural or in an artificially dried atmosphere, resist electrical conduction so strongly that they may support a body, or otherwise form a material communication between it and the earth, and yet allow it to remain charged with electricity to a potential sensibly differing from the earth's, for fractions of a second, for minutes, for hours, for days, or even for years, without any fresh excitation or continued source of electricity. Again, air, whether dry or saturated with vapour of water, and probably all gases and vapours, unless ruptured by too strong an electromotive force, are very thoroughly destitute of conductivity—that is to say, are very perfectly endowed with the property of resisting the tendency of electricity to pass and establish

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* Two conducting bodies are said to be of the same electric potential when, if put in conducting communication with the two electrodes of an electrometer, no electric effect is produced. When, on the other hand, the electrometer shows an effect, the amount of this effect measures the difference of potentials between the two bodies thus tested. Difference of potentials is also called electromotive force.

equality of potential between two bodies not otherwise materially connected.

250. Hence, when "the surface of the earth" is spoken of, the surface separating the solids and liquids of the earth from the air will be meant; and when the more qualified expression "outer surface of the earth" is used, inner surfaces of vesicles, or the surfaces bounding completely enclosed spaces of air, must be understood to be excluded. Thus, the surface of a mountain peak; the surface of a cave, up to the inmost recesses of the most intricate passages; the surface of a tunnel; the surface of the sea, or of a lake or river; all the surface of a sheet of unbroken water in such a fall as that of Niagara; the surface of blades of grass and flowers, and of soil below; in a wood, the surface of soil, and of trunks and leaves of trees; the surface of any animal resting on the earth; the outside of the roof of a house; the whole inside surface of a room with an open window; all belong to the outer surface of the earth.

251. On the other hand, the moon, meteoric stones, birds or insects flying, leaves or fruit falling, seed wafted through the air, spray breaking away from a cascade or from waves of the sea, the liquid particles of a cloud or a fog, present surfaces not belonging to the earth, and between which and the earth's surface differences of potential, and lines of electric force, may and generally do exist.

252. The whole surface of the earth, as defined above (§ 250), is at every moment electrified in every part, with the exception of neutral lines dividing portions which are negatively (resinously) from portions which are positively (vitreously) electrified. The negatively electrified portions are of very much greater extent, at all times, than those positively electrified; and there may be times when the whole surface is negatively electrified, because in all localities in which electrical observations have been hitherto made, with possibly one remarkable exception,* the earth's surface is always found negative, day and night,

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* At Guajara station, on the Peak of Teneriffe, "During the whole period "of observation, by day and night, the electricity was moderate in quantity, "and always resinous. This was during the period of N.E. trade wind, and "within its influence, though above its clouds."—[Professor Piazzi Smyth's Account of the Teneriffe Astronomical Experiment, *Philosophical Transactions*, 1858, and separate publication ordered by the Lords of the Admiralty.] The "electricity" here referred to was that acquired by an insulated conductor carrying a burning match in the air at some distance from the earth. If

during fair weather, and only occasionally positive in broken weather, or during an actual fall of rain in the immediate neighbourhood, if not exactly on the place of observation. If, then, at any one time there chances to be fair weather over the whole earth, it may be presumed that the whole outer surface of the earth is then negatively electrified, unless, judging from the possible exception above alluded to, we are still to expect positive electrification in same extreme positions.

253. As yet nothing is known regarding the electrification of air itself,* or of clouds or other matter suspended in the air, except what can be inferred (see below, § 254) from the electrification of the earth's surface, and its variations, with which alone, as Peltier has remarked, the observations of "atmospheric electricity" hitherto published have dealt (see below, §§ 265, 266). It is impossible, in the nature of things, to investigate the bodily electrification of a non-conductor by any observation whatever of electric action without it,† or in any way whatever, except by something equivalent to a determination of the magnitude and direction of the resultant force at every point of its mass.‡ Towards this thorough investigation

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it were really negative, the earth's electrification at the place must have been positive; but the test as to quality may have been deceptive, owing to the highly insulating condition of both outer and inner surfaces of the glass shade enclosing the gold leaves, and to the circumstance of the testing piece of rubbed sealing wax having been applied possibly too near the gold leaves, instead of beside a remote part of the insulated rod. Professor Smyth assures the writer, that he considers the electrical experiment as not sufficiently complete or confirmed to allow any conclusion to be built on it, and regards it rather as an indication of the importance of making electrical observations with better apparatus, and more available time for using it, than the first Teneriffe scientific expedition afforded.

* For knowledge gained since this article was written see §§ 296-301 below.

† According to Green's remarkable theorems, triply rediscovered by Gauss, Chasles, and the writer of this article, all different distributions of electricity within a solid, which produce the same potential at its surface, produce the same force at every point without it, and the problem of finding a distribution of electricity within the interior, to produce a given distribution of potential at the surface, is indeterminate.

‡ Let X, Y, Z be the components of the resultant force on a unit of electricity, if placed at any point x, y, z in a mass of air or other non-conductor; and let ρ denote the electrical density of the substance, that is to say, the quantity of electricity per unit of bulk actually possessed by the air in the neighbourhood of this point. Then, by a well-known proposition of the mathematical theory of attraction, we have

$$\rho = \frac{1}{4\pi}\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right).$$

of the distribution of electricity within a non-conducting mass; it may be remarked, that a determination of the normal component of the force all round a closed surface is just sufficient to show the aggregate quantity of electricity possessed by all the matter situated within it.* Hence observation in positions all round a mass of air is necessary for determining the quantity of electricity which it contains; and, therefore, the balloon must be put in requisition if knowledge of the distribution of electricity through the atmosphere is to be sought for.

254. Without leaving the earth, however, although we cannot thoroughly investigate the electrification of the air, we can make important inferences about it from observations of the electric density over the earth's surface, by a principle of judging which may be thus explained:—If the earth were simply an electrified body, placed in a perfectly insulating medium of indefinite extent, and not sensibly influenced by any other electrified matter, or by reflex influence from any conductor or dielectric in its vicinity, its electricity would be distributed over its surface according to a perfectly definite law, depending solely on the form of the surface, and deducible by a sufficiently powerful mathematical analysis from sufficiently perfect data of "geometry" (in the primitive sense of the term), or of what, in more modern language, is called geodesy. If the surface of the earth were truly spherical, this law would simply be uniform distribution. A truly elliptic oblateness of the earth would give, instead of uniformity, a distribution of electric density in simple proportion to the perpendicular distance between a tangent (that is horizontal) plane through any point and the earth's centre; according to which the electric density at the equator would be greatest, and would exceed that at either pole, where it would be least, by $\frac{1}{303}$: a difference which, for the present, we may disregard.

255. The whole amount of electricity over the surface of any great region of mountainous country, or of forest land, or of soil and vegetation of any kind, or of streets and houses in

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* Let N be the normal component of the force at any point of a closed surface, ds an element of the surface, $\int$ the sign of integration for the whole surface, and Q the whole quantity of electricity within it. Then, by a well-known theorem of Green's, rediscovered as alluded to in a preceding note, we have

$$Q = \frac{1}{4\pi}\int N ds.$$

a-town, or of rough sea, would be very approximately the same as that on an area of unruffled ocean, equal to the "reduced" area of the irregular surface; but the distribution of the electricity over hill and valley, over the leaves and trunks of trees, and the surfaces of plants generally, and on the soil beneath them, over the roofs, perpendicular walls, and overhanging or overshaded surfaces of buildings, and the surfaces of streets and enclosed courts between them, and over the hollows and crests of waves in a stormy sea, would be extremely irregular, with, in general, greater electric density on the more prominent and convex portions of surfaces, and less on the more covered and concave—quite insensible, indeed, in any such position as the interior of a cave, or the soil below trees in a forest even where considerable angular openings of sky are presented,—or the roof or floor of a tunnel, or covered chamber, even although open to a considerable angle of sky.

256. If thus a perfect electro-geodesy gave a "reduced" electric density equal over the whole earth, we might infer that the electrification of the earth is not influenced by any electricity in the air. According to what has been stated above, there might in that case be either no electricity in the air, from the earth's atmosphere to the remotest star, and the lines of electric force rising from the earth might either be infinite or terminate in the surfaces of the moon, meteoric stones, sun, planets, and stars; or there might be, at any distance considerably exceeding the height of the highest mountain, a uniformly electrified stratum of equal quantity and opposite kind to the earth's, balancing through all the exterior space the force due to the terrestrial electricity, and limiting the manifestations of electric force to the atmosphere within it; or there might be any of the infinite variety of distributions of electricity in space round the earth, by which the electric density at the earth's surface would be uninfluenced.

257. But, in reality, the electric density varies greatly, even in serene weather, over the earth's surface at any one time, as we may infer from (1.) the facts (established for Europe, and probably true in all the temperate zones of both hemispheres), that in any one place the electric density of the surface observed during serene weather is much greater in

winter than in summer, and that it varies according to something of a regular periodicity with the hours of the day and night; and (2.) the consideration that there is often serene weather of day and night, and of summer and winter, at one and the same time, in different temperate portions of the earth. We may, therefore, consider it as quite established that, even in serene weather, the electrification of the earth's surface is largely influenced by external electrified matter. Although we cannot (§ 253) discover the exact locality and distribution of this influencing electricity from its effects at the earth's surface alone, yet it is possible, from the character of the distribution of the terrestrial electric density as influenced by it, to assign a superior limit to its height.* If at any one instant the electric density reduced to the sea level were distributed according to a simple "harmonic" law, or, more generally, according to a certain definite character of non-abruptness of variation easily specified in mathematical language,† the external influencing electricity might be at any distance, however great, for all we could discover by observations near the earth's surface. But, little as we know yet regarding the diurnal law of electric variation in serene weather, it is, we may say with almost perfect certainty, not such as could give at any instant a distribution over the whole earth possessing any such gradual character as that referred to; and, therefore, we may, in all probability, from the character of the diurnal variation itself, say that its electric origin is not at a distance of many radii from the surface. On the other hand, when we consider that in temperate regions the velocity with which the earth's surface is carried round in its diurnal course is from 500 to 900 miles per hour, we see clearly that any law of diurnal electric variation, established on observations even so frequent as once every hour, could not possibly fix the locality of the origin to within 100 miles of the surface; and as we have as yet nothing to go upon in the way of published observations more frequent than three or four times a day, towards establishing either the existence or the character of the diurnal law, we cannot consider it as proved by observation that the influencing electricity which produces it is even as near as the 50 or 100 miles limit which is commonly (but in the opinion of the writer of this article, most unreasonably) assigned as an end to the earth's atmosphere.

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* If at any instant the co-efficients of the series of "Laplace's functions," expressing the terrestrial electric density reduced to the sea level, converged ultimately with less rapidity than the geometrical series $1, \frac{1}{m}, \frac{1}{m^2}, \ldots$ we might be sure that there is electricity in the air at some distance from the centre of the earth, not exceeding m times the radius of the earth's surface. For the principles on which this assertion is founded, see a short article, entitled "Note on Certain Points in the Theory of Heat," *Cambridge Mathematical Journal*, November 1843.

† For instance, if in simple proportion to the cosine of the angular distance from any point of the earth's surface, or more generally, if expressible by any finite number of "Laplace's functions," or still more generally, if expressible by a series of "Laplace's functions," with co-efficients converging ultimately more rapidly than any geometrical series.

258. The great suddenness of the electric variations during broken weather, and their close correspondence with beginnings, changes, and cessations of rain, hail, or snow, compel us (by a *common sense* estimate founded on an unconscious application of the mathematical law stated in the footnotes to the preceding § 257) to believe that their origin agrees in position with that of the showers, and to give it a "local habitation" and a name—Thundercloud.

259. The writer of this article has observed extremely rapid variations of terrestrial electrification during perfectly serene weather. Thus, in a calm summer night, with an unvarying cloudless sky overhead, and not the faintest appearance of auroral light to be seen, he has, in a temporary electric observatory in the Island of Arran, found large variations (as much as from a certain degree to double and back) in the course of a minute of time. The influencing electricity by which these variations were produced, cannot possibly (unless on the extremely improbable hypothesis of their being due to highly electrified extra-terrestrial matter moving very rapidly with reference to the earth) have been very far removed from the earth's surface. It is not impossible, and we have as yet nothing to make it decidedly improbable, that they were due to fluctuations up and down of aërial strata, perhaps those of the great atmospheric currents, in high regions of the atmosphere. Judging, however, from still more recent observations referred to below (§ 262), we may think it more probable that these remarkable variations in the observed electric force were

due chiefly to positively or negatively electrified masses moving along within a few miles of the locality of observation.

260. Returning to the subject of the distribution of electricity over the earth's surface at any instant, we may remark, that if over an area of several miles in diameter, of perfectly level bare country, or of sea, the electrical density is sensibly uniform, we could not, without going up in a balloon, and observing the electric force at points in the air above, form any judgment whatever as to the distance from the earth at which the influencing electricity is situated. If, on the other hand, we find a very sensible variation in the electric density between two points of a piece of level open country, or at sea, not many miles apart, we may infer as quite certain that there is influencing electricity not many miles up in the air, and not uniformly distributed in level strata. Nothing can be easier than to make this trial—only to observe simultaneously with similar instruments, similarly placed, at two neighbouring stations, in a suitable locality—and most interesting and important results are to be derived from it, as soon as arrangements can be made for continuing the requisite observations day and night, during various vicissitudes of weather, especially during a time of perfect serenity.

261. Corresponding statements apply to a mountainous country, with this modification, that a very varied, instead of a uniform distribution of electric density, is, in such a locality, as explained above in § 255, the natural consequence of freedom from the disturbing influence of near electrified masses of air or cloud. The problem of accurately determining, from purely geometric data (§ 256), this undisturbed distribution over even the smoothest hillside, would infinitely transcend human mathematical power, although an approximate solution may be readily given for any piece of country over the whole of which both the inclination and the ratio of the height above the general level to the radius of curvature of the surface are small. For a rugged mountainous country, the most perfect geometric data, and the most strenuous mathematical efforts, could scarcely lead us towards an approximate estimate of the inequalities of electric density which different localities must present without any disturbance from near electrified atmosphere. Hence, in a

mountainous country—unless we find electricity strong in some locality where from the configuration of the surface, we correctly judge it ought to be weak if undisturbed, or weak where it ought to be strong, or unless, at least, we find some very decided deviation from any such amount of difference between two stations as, without being able to make a precise calculation, we can estimate for the difference due to figure—we cannot judge as to the influence of aërial electrification from simultaneous absolute determinations at any one instant alone. But of one thing we may be sure, that although the absolute amounts of the electrification at any two stations not far apart may differ largely, they must remain in an absolutely constant proportion to one another, if there is no electrified air or cloud near.

262. Hence, if we find observations made simultaneously by two electrometers in neighbouring positions, in a mountainous country, to bear always the same mutual proportion, we may not be able to draw any inference as to electrified air; but if, on the contrary, we find their proportion varying, we may be perfectly certain that there are varying electrified masses of air or cloud not far off. A first application of this test is described in the following extract from the Proceedings of the Literary and Philosophical Society of Manchester for October 18, 1859:—

"The following extract of a letter received from Professor W. "Thomson, F.R.S., Glasgow, Honorary Member of the Society, "etc., was read by Dr. Joule:—

'I have a very simple "domestic" apparatus by which I can 'observe atmospheric electricity in an easy way. It consists 'merely of an insulated can of water set on a table or window 'sill *inside*, and discharging by a small pipe through a fine nozzle 'two or three feet from the wall. With only about ten inches 'head of water and a discharge so slow as to give no trouble in 'replenishing the can with water, the atmospheric effect is 'collected so quickly that any difference of potentials between 'the insulated conductor and the air at the place where the 'stream from the nozzle breaks into drops is done away with at 'the rate of five per cent. per half second, or even faster. Hence 'a very moderate degree of insulation is sensibly as good as 'perfect, so far as observing the atmospheric effect is concerned. 'It is easy, by my plan of drying the atmosphere round the 'insulating stems by means of pumice-stone moistened with

‘ sulphuric acid, to insure a degree of insulation in all weathers, ‘ by which there need not be more than five per cent. per hour ‘ lost by it from the atmospheric apparatus at any time. A little ‘ attention to keep the outer part of the conductor clear of ‘ spider lines is necessary. The ‘ apparatus I employed at In‘vercloy stood on a table beside ‘ a window on the second floor, ‘ which was kept open about ‘ an inch to let the discharging tube project out without ‘ coming in contact with the ‘ frame. The nozzle was only about two feet and a half from ‘ the wall, and nearly on a level with the window sill. The ‘ divided ring electrometer stood on the table beside it, and ‘ acted in a very satisfactory way (as I had supplied it with a ‘ Leyden phial, consisting of a common thin white glass shade, ‘ which insulated remark‘ably well, instead of the ‘ German glass jar — the ‘ second of the kind which ‘ I had tried, and which ‘ would not hold its charge ‘ for half a day). I found ‘ from 13¼° to 14° of torsion ‘ required to bring the index ‘ to zero, when urged aside ‘ by the electromotive force ‘ of ten zinc-copper water ‘ cells. The Leyden phial ‘ held so well, that the sensi‘bility of the electrometer, ‘ measured in that way, did ‘ not fall more than from ‘ 13½° to 13¼° in three days. ‘ The atmospheric effect ‘ ranged from 30° to above ‘ 420° during the four days ‘ which I had to test it; that

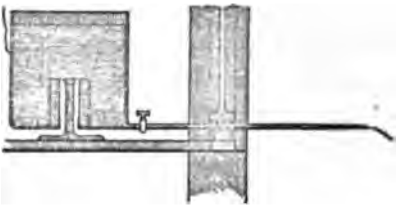

FIG. 1.

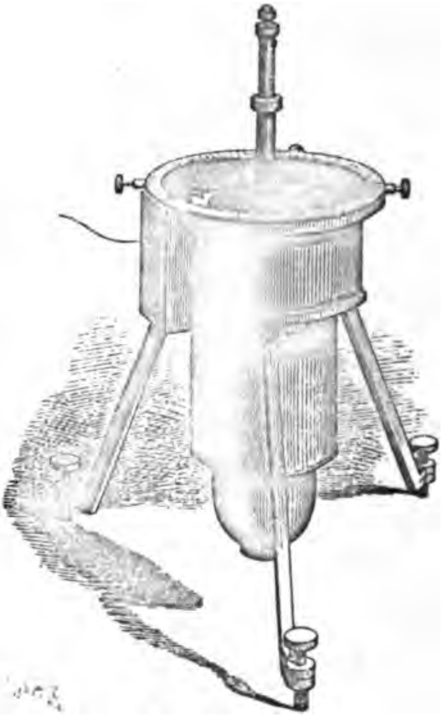

FIG. 2.

'is to say, the electromotive force per foot of air, measured horizontally from the side of the house, was from 9 to above 126 zinc-'copper water cells. The weather was almost perfectly settled, 'either calm, or with slight east wind, and in general an easterly 'haze in the air. The electrometer twice within half an hour went 'above 420°, there being at the time a fresh temporary breeze 'from the east. What I had previously observed regarding the 'effect of east wind was amply confirmed. Invariably the 'electrometer showed very high positive in fine weather, before 'and during east wind. It generally rose very much shortly 'before a slight puff of wind from that quarter, and continued 'high till the breeze would begin to abate. I never once 'observed the electrometer going up unusually high during fair 'weather without east wind following immediately. One evening in August I did not perceive the east wind at all, when 'warned by the electrometer to expect it; but I took the 'precaution of bringing my boat up to a safe part of the beach, 'and immediately found by waves coming in that the wind 'must be blowing a short distance out at sea, although it did 'not get so far as the shore. I made a slight commencement 'of the *electrogeodesy* which I pointed out as desirable at the 'British Association, and in the course of two days, namely, 'October 10th and 11th, got some very decided results. Macfarlane, and one of my former laboratory and Agamemnon 'assistants, Russell, came down to Arran for that purpose. Mr. 'Russell and I went up Goatfell on the 10th instant, with the 'portable electrometer (see Fig. 3), and made observations, while 'Mr. Macfarlane remained at Invercloy, constantly observing 'and recording the indications of the house electrometer. On 'the 11th instant the same process was continued, to observe 'simultaneously at the house and at one or other of several 'stations on the way up Goatfell. I have not yet reduced all 'the observations; but I see enough to leave no doubt whatever 'but that cloudless masses of air at no great distance from the 'earth, certainly not more than a mile or two, influence the 'electrometer largely by electricity which they carry. This I 'conclude because I find no constancy in the relation between 'the simultaneous electrometric indications at the different 'stations. Between the house and the nearest station the rela-

'tive variation was least. Between the house and a station about 'half way up Goatfell, at a distance estimated at two miles and 'a half in a right line, the number expressing the ratio varied 'from about 113 to 360 in the course of about three hours. On 'two different mornings the ratio of the house to a station about 'sixty yards distant on the road beside the sea was 97 and 96 'respectively. On the afternoon of the 11th instant, during a 'fresh temporary breeze of east wind, blowing up a little spray as 'far as the road station, most of which would fall short of the 'house, the ratio was 108 in favour of the house electrometer '—both standing at the time very high—the house about 350°. 'I have little doubt but that this was owing to the negative 'electricity carried by the spray from the sea, which would 'diminish relatively the indications of the road electrometer.'"

263. The electrometers referred to in the preceding extract were on two different plans. The first, or "divided ring electrometer," consists of—(1.) A ring of metal divided into sectors, of which some—one or more—are insulated and connected with the conductor to be electrically tested, and the remainder connected with the earth. (2.) An index of metal supported by a glass fibre, or a wire, stretched in the line of the axis of the ring, and capable of having its fixed end turned through angles measured by a circle and pointer. (3.) A Leyden phial, with its insulated coating electrically connected with the index. (4.) A case to protect the index from currents of air, and to keep an artificially dried atmosphere round the insulating supports—glazed to allow the index to be seen from without, but with the inner surface of the glass screened (electrically) by wire cloth, perforated metal, or tinfoil, to do away with irregular reflections on the index. In the instrument represented in the drawing (No. 2) above, the ring is divided only into two parts, which are equal, and separated by a space of air about $\frac{1}{10}$ of an inch. Each of these half rings is supported on two glass pillars; and by means of screws acting on a foot which bears these pillars, it is adjusted and fixed in its proper position. The index is of thin sheet aluminium, and projects in only one direction from the glass fibre bearing it. A stiff vertical wire, rigidly connected with it, nearly in the prolongation of the fibre, bears a counterpoise considerably

below the level of the index, and heavy enough to keep the index horizontal. A thin platinum wire hooked to the lower end of this vertical wire, dips in sulphuric acid in the bottom of the Leyden phial. The Leyden phial is charged either positively or negatively; and is found to retain its charge for months, losing, however, gradually, at some low rate, less generally than one per cent. per day of its amount. The index is thus, when the instrument is in use, kept in a state of charge corresponding to the potential of the inside coating of the phial. When one of the half rings is connected with the earth, and a charge of electricity communicated to the other, the index moves from or towards the latter, according as the charge communicated to it is of the same or the opposite kind to that of the index. This instrument, as an electroscope, possesses extreme sensibility—much greater than that of any other hitherto constructed; and by the aid of the torsion arrangement, it may be made to give accurate metrical results. There are some difficulties in the use of it, especially as regards the comparison of the indications obtained with different degrees of electrification of the index, and the reduction of the results to absolute measure, hitherto obviated only by a daily application of Delmann's method of reference to a zinc-copper water battery, which Delmann himself applies once for all, to one of his electrometers (unless his glass fibre breaks, when he must make a fresh determination of the sensibility of the instrument with its new fibre). The high sensibility of the divided ring

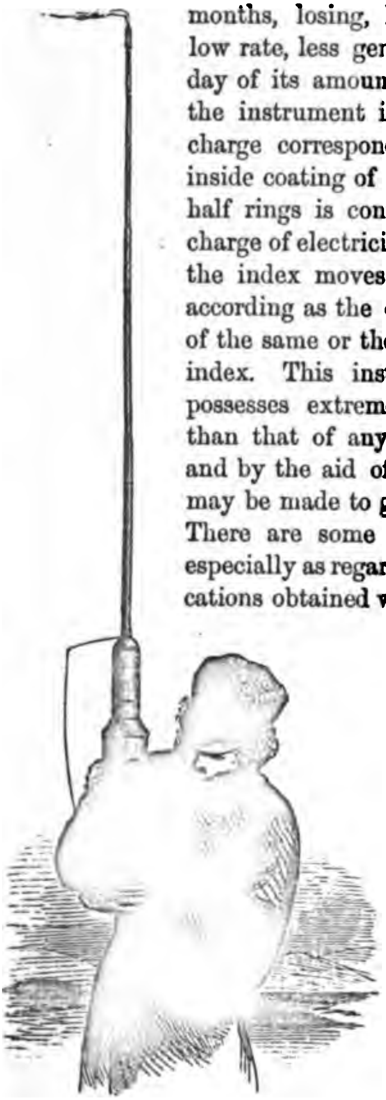

Fig. 2.—Portable Atmospheric Electrometer.

electrometer renders this test really very easy, as not more than from ten to twenty cells are required; and a comparison with a few good cells of Daniell's may be made by its aid, to ascertain the absolute value and the constancy of the water cells. The difficulty thus met is altogether done away with in another kind of electrometer, also "heterostatic," of which only one has yet been constructed—the electrometer of the portable apparatus shown in the third drawing. In it the index is attached at right angles to the middle of a fine platinum wire, firmly stretched between the inside coatings of two Leyden phials, and consists simply of a very light bar of aluminium, extending equally on the two sides of the supporting wire. It is repelled by two short bars of metal, fixed on the two sides of the top of a metal tube, which is supported by the inside coating of the lower phial, and has the fine wire in its axis. A conductor of suitable shape, bearing an electrode, to connect with the body to be tested, insulated inside the case of the instrument, in the neighbourhood of the index, and when electrified in the same way, or the contrary way, to the inside coatings of the Leyden phials, causes, by its influence, the repulsion between the index and the fixed bars to be diminished or increased. The upper Leyden phial is moveable about a fixed axis, through angles measured by a pointer and circle, and thus the amount of torsion, in one-half of the bearing wire, required to bring the index to a constant position, in any case, is measured. The square root of the number of degrees of torsion measures the difference of potentials between the conductor tested and the inner coating of the Leyden phial. In using the instrument, the conductor tested is first put in connexion with the earth, and the torsion required to bring the index to its fixed position is read off. This is called the zero, or earth reading. The tested conductor is then electrified, and the torsion reading taken. In the atmospheric application, this is called the air reading. The excess—positive or negative—of its square root, above that of the zero reading, measures the electromotive force between the earth and the point of air tested. This result, when positive shows vitreous, when negative resinous potential in the air; if the index is resinous. By the aid of Barlow's table of square roots, the indications of the instrument may thus be reduced to definite measure of potential, almost as quickly as they can be written down. Once for all, the sensibility of the instrument can be determined by comparison with an absolute electrometer, or a galvanic battery. In the portable apparatus a burning match is used—instead of the water-dropping system, which the writer finds more convenient than any other for a fixed apparatus—to reduce the insulated conductor to the same potential as the air at its end.

264. As has been remarked above (§ 252), it is the electrification of the earth's surface which has either directly or virtually been the subject of measurement in all observations on atmospheric electricity hitherto made. The methods which have been followed may be divided into two classes—(1.) Those in which means are taken to reduce the potential of an insulated conductor to the same as that of the air, at some point, a few feet or yards distant from the earth. (2.) Those in which a portion of the earth (see above, § 253) is insulated, removed from its position, and tested by an electrometer, in a different position, or under cover. The first method was very imperfectly carried out by Beccaria with his long "exploring wire," stretched between insulating supports, or elevated portions of buildings, tree tops, or other prominent positions of the earth (see above, § 249); also, very imperfectly by means of "Volta's lantern"—an enclosed flame, supported on the top of an insulated conductor. On the other hand, it is put in practice very perfectly, by means of a match, or flame burning in the open air, on the top of a well insulated conductor—a plan adopted, after Volta's suggestion, by many observers; also, even more decidedly, by means of the water-dropping system—described in the preceding extract—which has recently occurred to the writer, and has been found by him both to be very satisfactory in its action, and extremely easy and convenient in practice. The principle of each of these methods of the first class may be explained best by first considering the methods of the second class, as follows:—

265. If a large sheet of metal were laid on the earth in a perfectly level district, and if a circular area of the same metal were laid upon it, and, after the manner of Coulomb's proof plane, were lifted by an insulated handle, and removed

to an electrometer within doors, a measure of the earth's electrification, at the time, would be obtained; or, if a ball, placed on the top of a conducting rod in the open air, were lifted from that position by an insulating support, and carried to an electrometer within doors, we should also have, on precisely the same principle, a measure of the earth's electrification at the time. If the height of the ball in this second plan were equal to one-sixteenth of the circumference of the disc (compare § 235) used in the first plan, the electrometric indications would be the same, provided the diameter of the ball is small, in comparison with the height to which it is raised in the air, and the electrostatic capacity of the electrometer is small enough not to take any considerable proportion of the electricity from the ball in its application. The idea of experimenting by means of a disc laid flat on the earth, is merely suggested for the sake of illustration, and would obviously be most inconvenient in practice. On the other hand, the method, by a carrier ball, instead of a proof plane, is precisely the method by which, on a small scale, Faraday investigated the distribution of electricity induced on the earth's surface (see above, § 249), by a piece of rubbed shell-lac; and the same method, applied on a suitable scale, for testing the natural electrification of the earth in the open air, has given, in the hands of Delmann of Creuznach, the most accurate results hitherto published in the way of electro-meteorological observation.*

266. If, now, we conceive an elevated conductor, first belonging to the earth (§ 249), to become insulated, and to be made to throw off, and to continue throwing off, portions from an exposed position of its own surface, this part of its surface will quickly be reduced to a state of no electrification, and the whole conductor will be brought to such a potential as will allow it to remain in electrical equilibrium in the air, with that portion of its surface neutral. In other words, the potential throughout the insulated conductor is brought to be the same as that of the

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* Through some misapprehension, Mr. Delmann himself has not perceived that his own method of observation really consists in removing a portion of the earth, and bringing it insulated with the electricity which it possessed *in situ*, to be tested within doors, otherwise, he could not have objected, as he has, to Peltier's view.

particular equi-potential surface in the air, which passes through the point of it from which matter breaks away. A flame, or the heated gas passing from a burning match, does precisely this: the flame itself, or the highly-heated gas close to the match being a conductor which is constantly extending out, and gradually becoming a non-conductor. The drops into which the jet issuing from the insulated conductor, on the plan introduced by the writer, produce the same effects, with more pointed decision, and with more of dynamical energy to remove the rejected matter with the electricity which it carries from the neighbourhood of the fixed conductor.

ROYAL INSTITUTION FRIDAY EVENING LECTURE,

MAY 18, 1860.

267. Stephen Gray, a pensioner of the Charter-house, after many years of enthusiastic and persevering devotion to electric science, closed his philosophical labours, about one hundred and thirty years ago, with the following remarkable conjecture:—"That there may be found a way to collect a greater "quantity of the electrical fire, and consequently to increase "the force of that power, which, by several of these experi"ments, *si licet magna componere parvis*, seems to be of the "same nature with that of thunder and lightning."

The inventions of the electrical machine and the Leyden phial immediately fulfilled these expectations as to collecting greater quantities of electric fire; and the surprise and delight which they elicited by their mimic lightnings and thunders, and above all by the terrible electric shock, had scarcely subsided when Franklin sent his kite messenger to the clouds, and demonstrated that the imagination had been a true guide to this great scientific discovery—the identity of the natural agent in the thunderstorm with the mysterious influence produced by the simple operation of rubbing a piece of amber, which, two thousand years before, had attracted the attention of those philosophers among the ancients who did not despise the small things of nature.

268. The investigation of atmospheric electricity immediately became a very popular branch of natural science; and the discovery of remarkable and most interesting phenomena quickly rewarded its cultivators. The foundation of all we now know was completed by Beccaria, in his observations on "the mild electricity of serene weather," nearly a hundred years ago. It was not until comparatively recent years that definite quantitative comparisons from time to time of the electric quality manifested by the atmosphere in one locality were first obtained by the application of Peltier's mode of observation with his metrical electroscope. The much more accurate electrometer, and the greatly improved mode of observation, invented by Delmann, have given for the electric intensity, at any instant, still more precise results; but have left something to desire in point of simplicity and convenience for general use, and have not afforded any means for continuous observation, or for the introduction of self-recording apparatus. The speaker had attempted to supply some of these wants, and he explained the construction and use of instruments, now exhibited to the meeting, which he had planned for this purpose.

269. Apparatus for the observation of atmospheric electricity has essentially two functions to perform; to electrify a body with some of the natural electricity, or with electricity produced by its influence; and to measure the electrification thus obtained.

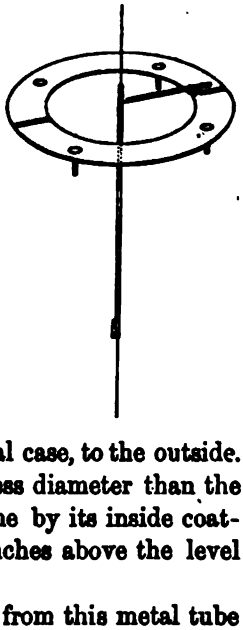
270. The measuring apparatus exhibited, consisted of three electrometers, which were referred to under the designations of (I.) The divided ring reflecting electrometer; (II.) The common house electrometer; and (III.) The portable electrometer.

(I.) The divided ring reflecting electrometer [compare § 263, above, and §§ 444...456, below] consists of:—

(1.) A ring of metal divided into two equal parts, of which one is insulated, and the other connected with the metal case (5.) of the instrument.

(2.) A very light needle of sheet aluminium hung by a fine glass fibre, and counterpoised so as to make it project only to one side of this axis of suspension.

(3.) A Leyden phial, consisting of an open glass jar, coated outside and inside in the usual manner, with the exception that the tinfoil of the inner coating does not extend to the bottom of the jar, which is occupied instead by a small quantity of sulphuric acid [connected with the tinfoil by means of a platinum wire].



(4.) A stiff straight wire rigidly attached to the aluminium needle, as nearly as may be in the line of the suspending fibre, bearing a light platinum wire linked to its lower end, and hanging down so as to dip into the sulphuric acid.

(5.) A case protecting the needle from currents of air, and from irregular electric actions, and maintaining an artificially dried atmosphere round the glass pillar or pillars supporting the insulated half-ring and the uncoated portion of the glass of the phial.

(6.) A light stiff metallic electrode projecting from the insulated half-ring through the middle of a small aperture in the metal case, to the outside.

(7.) A wide metal tube of somewhat less diameter than the Leyden jar, attached to a metal ring borne by its inside coating, and standing up vertically to a few inches above the level of the mouth of the jar.

(8.) A stiff wire projecting horizontally from this metal tube above the edge of the Leyden jar, and out through a wide hole in the case of the instrument to a convenient position for applying electricity to charge the jar with.

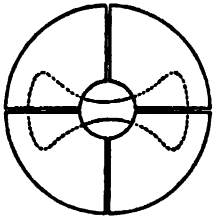
(9.) A very light glass mirror, about three-quarters of an inch diameter, attached by its back to the wire (4.), and therefore rigidly connected with the aluminium needle.

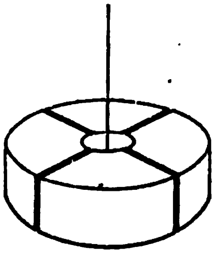
(10.) A circular aperture in the case shut by a convex lens, and a long horizontal slit shut by plate glass, with its centre immediately above or below that of the lens, one of them above, and the other equally below the level of the centre of the mirror.

(11.) A large aperture in the wide metal tube (7.), on a level with the mirror (9.), to allow light from a lamp outside the case, entering through the lens, to fall upon the mirror, and be reflected out through the plate-glass window; and three or four fine metal wires stretched across this aperture to screen the mirror from irregular electric influences, without sensibly diminishing the amount of light falling on and reflected off it.

271. The divided ring (1.) is cut out of thick strong sheet metal (generally brass). Its outer diameter is about 4 inches, its inner diameter 2½; and it is divided into two equal parts by cutting it along a diameter with a saw. The two halves are fixed horizontally; one of them on a firm metal support, and the other on glass, so as to retain as nearly as may be their original relative position, with just the saw cut, from $\frac{1}{16}$ to $\frac{1}{20}$ of an inch broad, vacant between them. They are placed with their common centre as nearly as may be in the axis of the case (5.), which is cylindrical, and placed vertically. The Leyden jar (3.), and the tube (7.), carried by its inside coating, have their common axis fixed to coincide as nearly as may be with that of the case and divided ring. The glass fibre hangs down from above in the direction of this axis, and supports the needle about an inch above the level of the divided ring. The stiff wire (4.), attached to the needle, hangs down as nearly as may be along the axis of the tube (7.).

[The following diagrams, placed here to facilitate comparison, represent the arrangement of "needle" and quadrants described below in § 345, as substituted in the modern instrument for the bisected ring and narrow needle of the old electrometer here described]:—





272. Before using the instrument, the Leyden phial (3.) is charged by means of its projecting electrode (8.). When an electrical machine is not available, this is very easily done by the aid of a stick of vulcanite, rubbed by a piece of chamois leather. The potential of the charge thus communicated to the phial, is

to be kept as nearly constant as is required for the accuracy of the investigation for which the instrument is used. Two or three rubs of the stick of vulcanite once a day, or twice a day, are sufficient when the phial is of good glass, well kept dry. The most convenient test for the charge of the phial is a proper electrometer or electroscope, of any convenient kind, kept constantly in communication with the charging electrode (8.). [Compare § 353, below.]

The electrometer (II.) is to be ordinarily used for that purpose in the Kew apparatus. Failing any such gauge electrometer or electroscope, a zinc-copper-water battery of ten, twenty, or more small cells may be very conveniently used (after the manner of Delmann) to test directly the sensibility of the reflecting electrometer, which is to be brought to its proper degree by charging its Leyden phial as much as is required.

273. In the use of this electrometer, the two bodies of which the difference of potentials is to be tested are connected, one of them, which is generally the earth, with the metal case of the instrument, and the other with the insulated half ring. The needle being, let us suppose, negatively electrified, will move towards or from the insulated half ring, according as the potential of the conductor connected with this half ring differs positively or negatively from that of the other conductor (earth) connected with the case. The mirror turns accordingly in one direction or the other through a small angle from its zero position, and produces a corresponding motion in the image of the lamp on the screen on which it is thrown.

274. (II.) The common house electrometer [compare § 263, above, and §§ 374...377, below].—This instrument consists of:—

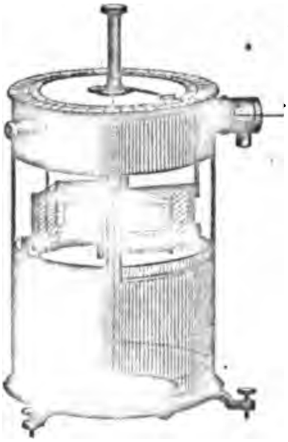
(1.) A thin flint-glass bell, coated outside and inside like a Leyden phial, with the exception of the bottom inside, which contains a little sulphuric acid.

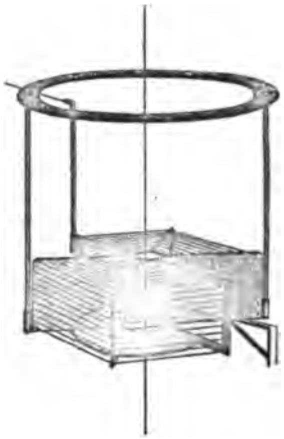
(2.) A cylindrical metal case, enclosing the glass jar, cemented to it round its mouth outside, extending upwards about an inch and a half above the mouth, and downwards to a metal base supporting the whole instrument, and protecting the glass against the danger of breakage.

(3.) A cover of plate glass, with a metal rim, closing the top of the cylindrical case of the instrument.

(4.) A torsion head, after the manner of Coulomb's balance, supported in the centre of the glass cover, and bearing a glass fibre which hangs down through an aperture in its centre.

(5.) A light aluminium needle attached across the lower end of the fibre (which is somewhat above the centre of the glass bell), and a stiff platinum wire attached to it at right angles, and hanging down to near the bottom of the jar.





(6.) A very light platinum wire, long enough to hang within one-eighth of an inch or so of the bottom of the jar, and to dip in the sulphuric acid.

(7.) A metal ring, attached to the inner coating of the jar, bearing two plates in proper positions for repelling the two ends of the aluminium needle when similarly electrified, and proper stops to limit the angular motion of the needle to within about 45° from these plates.

(8.) A cage of fine brass wire, stretched on brass framework, supported from the main case above by two glass pillars, and partially enclosing the two ends of the needle, and the repelling plates, from all of which it is separated by clear spaces, of nowhere less than one-fourth of an inch of air.

(9.) A charging electrode, attached to the ring (7.), and projecting over the mouth of the jar to the outside of the metal case (2.), through a wide aperture, which is commonly kept closed by a metal cap, leaving at least one-quarter of an inch of air round the projecting end of the electrode.

(10.) An electrode attached to the cage (8.), and projecting over the mouth of the jar to the outside of the metal case (2.), through the centre of an aperture, about a quarter of an inch diameter.

274. This instrument is adapted to measure differences of potential between two conducting systems, namely; as one, the aluminium needle (5.), the repelling plates (7.), and the inner coating of the jar; and, as the other, the insulated cage (8.). This latter is commonly connected by means of its projecting electrode (10.), with the conductor to be tested. The two conducting systems, if through their projecting electrodes connected by a metallic wire, may be electrified to any degree, without causing the slightest sensible motion in the needle. If, on the other hand, the two electrodes of these two systems are connected with two conductors, electrified to different potentials, the needle moves away from the repelling plates; and if, by turning the torsion head, it is brought back to one accurately marked position, the number of degrees of torsion required is proportional to the square of the difference of potentials thus tested.

275. In the ordinary use of the instrument, the inner coating of the Leyden jar is charged negatively, by an external application of electricity through its projecting electrode (9.). The degree of the charge thus communicated, is determined by putting the cage in connexion with the earth through its electrode (10.), and bringing the needle by torsion to its marked position. The square root of the number of degrees of torsion required to effect this, measures the potential of the Leyden charge. This result is called the reduced earth reading. When the atmosphere inside the jar is kept sufficiently dry,—this charge is retained from day to day with little loss; not more, often, than one per cent. in the twenty-four hours.

276. In using the instrument the charging electrode (9.) of the jar is left untouched, with the aperture through which it projects closed over it by the metal cap referred to above. The electrode (10.) of the cage, when an observation is to be made, is connected with the conductor to be tested, and the needle is brought by torsion to its marked position. The square root of the number of degrees of torsion now required measures the difference of potentials between the conductor tested and the interior coating of the Leyden jar. The excess, positive or negative, of this result above the reduced earth reading, measures

the excess of the potential, positive or negative, of the conductor tested above that of the earth; or simply the potential of the conductor tested, if we regard that of the earth as zero.

277. (III.) The portable electrometer [compare § 263, above, and §§ 363...373, below] is constructed on the same electrical principles as the house electrometer just described. The mode of suspension of the needle is, however, essentially different; and a varied plan of connexion between the different electrical parts has been consequently adopted as more convenient. In the portable electrometer, the needle is firmly attached at right angles to the middle of a fine platinum wire, tightly stretched in the axis of a brass tube with apertures in its middle to allow the needle to project on the two sides. One end of the platinum wire is rigidly connected with this tube; the other is attached to a graduated torsion head. The brass tube carries two metal plates in suitable positions to repel the two ends of the needle in contrary directions, and metal stops to limit its angular motion within a convenient range. The conducting system composed of these different parts is supported from the metal cover, or roof of the jar, by three glass stems. The torsion head is carried round by means of a stout glass bar, projecting down from a pinion centered on the lower side of this cover, and turned by the action of a tangent screw presenting a milled head, to the hand of the operator outside. The conducting system thus borne by insulating supports is connected with the outside conductor to be tested by means of an electrode passing out through the centre of the top of the case by a wide aperture in the centre of the pinion. A wire cage, surrounding the central part of the tube and the needle and repelling plates, is rigidly attached to the interior coating of the Leyden jar. It carries two metal sectors, or "bulkheads," in suitable positions to attract the two ends of the needle, which, however, is prevented from touching them by the limiting stops referred to above. The effect of these attracting plates, as they will be called, is to increase very much the sensibility of the instrument. The square root of the number of degrees of torsion required to bring the needle to a sighted position near the repelling plates, measures the

difference of potentials between the cage and the conducting system, consisting of tube, torsion-head, repelling plates, and needle. The metal roof of the jar is attached to a strong metal case, cemented round the outside of the top of the jar, and enclosing it all round and below, to protect it from breakage when being carried about. There are sufficient apertures in this case, opened by means of a sliding piece, to allow the observer to see the needle and graduated circle (torsion head), when using the instrument. On the outside of the roof of the jar a stout glass stem is attached, which supports a light stiff metallic conductor, by means of which a burning match is supported, at the height of two or three feet above the observer. This conductor is connected by means of a fine wire with the electrometer, in the manner described above, through the centre of the aperture in the roof. An artificially dried atmosphere is maintained around this glass stem, by means of a metal case surrounding it, and containing receptacles of gutta percha, or lead, holding suitably shaped pieces of pumice-stone moistened with sulphuric acid. The conductor which bears the match projects upwards through the centre of a sufficiently wide aperture, and bears a small umbrella, which both stops rain from falling into this aperture, and diminishes the circulation of air, owing to wind blowing round the instrument, from taking place, to so great a degree as to do away with the dryness of the interior atmosphere required to allow the glass stem to insulate sufficiently. The instrument may be held by the observer in his hand in the open air without the assistance of any fixed stand. A sling attached to the instrument and passing over his left shoulder, much facilitates operations, and renders it easy to carry the apparatus to the place of observation, even if up a rugged hill side, with little risk of accident.

278. The burning match in the apparatus which has just been described, performs the collecting function referred to above. The collector employed for the station apparatus, whether the reflecting electrometer or the common house electrometer is used, is an insulated vessel of water, allowed to flow out in a fine stream through a small aperture at the end of a pipe projecting to a distance of several feet from the wall of the building in which the observations are made.

279. The principle of collecting, whether by fire or by water, in the observation of atmospheric electricity, was explained by the speaker thus:—The earth's surface is, except at instants, always found electrified, in general negatively, but sometimes positively. [Quotation from Nichol's *Cyclopædia*, viz., § 265, above, comes here in the original.]

After having given so much of these explanations as seemed necessary to convey a general idea of the principles on which the construction of the instruments of investigation depended, the speaker proceeded to call attention to the special subject proposed for consideration this evening.

280. What is terrestrial atmospheric electricity? Is it electricity of earth, or electricity of air, or electricity of watery or other particles in the air? An endeavour to answer these questions was all that was offered; abstinence from speculation as to the origin of this electric condition of our atmosphere, and its physical relations with earth, air, and water, having been painfully learned by repeated and varied failure in every attempt to see beyond facts of observation. In serene weather, the earth's surface is generally, in most localities hitherto examined, found negatively or resinously electrified; and when this fact alone is known, it might be supposed that the globe is merely electrified as a whole with a resinous charge, and left insulated in space.

281. But it is to be remarked that the earth, although insulated in its atmospheric envelope, being in fact a conductor, touched only by air one of the best although not the strongest of insulators, cannot with its atmosphere be supposed to be insulated so as to hold an electric charge in interplanetary space. It has been supposed, indeed, that outside the earth's recognised atmosphere there exists something or nothing in space which constitutes a perfect insulator; but this supposition seems to have no other foundation than a strange idea that electric conductivity is a strength or a power of matter rather than a mere *non-resistance*. In reality we know that air highly rarefied by the air-pump, or by other processes, as in the construction of the "vacuum tubes," by which such admirable phenomena of electric light have recently been seen in this place, becomes extremely weak in its resistance to the transference of elec-

tricity through it, and begins to appear rather as a conductor than an insulator. One hundred miles or upwards from the earth's surface, the air in space cannot in all probability have resisting power enough to bear any such electric forces as those which we generally find even in serene weather in the lower strata. Hence we cannot, with Peltier, regard the earth as a resinously charged conductor, insulated in space, and subject only to accidental influences from temporary electric deposits in clouds, or air round it; but we must suppose that there is always *essentially* in the higher aërial regions a distribution arising from the self-relief of the outer highly rarefied air by disruptive discharge. This electric stratum must constitute very nearly the electro-polar complement to all the electricity that exists on the earth's surface, and in the lower strata of the atmosphere; in other words, the total quantity of electricity, reckoned as excess of positive above negative, or of negative above positive, in any large portion of the atmosphere, and on the portion of the earth's surface below it, must be very nearly zero. The quality of non-resistance to electric force of the thin interplanetary air being duly considered, we might regard the earth, its atmosphere, and the surrounding medium as constituting respectively the inner coating, the di-electric (as it were glass), and the outer coating of a great Leyden phial, charged negatively; and even if we were to neglect the consideration of possible deposits of electricity through the body of the di-electric itself, we should arrive at a correct view of the electric indications discoverable at any one time and place of the earth's surface. In fact, any kind of "collector," or plan for collecting electricity from or in virtue of the natural "terrestrial atmospheric electricity," gives an effect simply proportional to the electrification of the earth's surface then and there. The methods of collecting by fire and water which the speaker exhibited, gave definitively, in the language of the mathematical theory, the "electric potential" of the air at the point occupied by the burning end of the match, or by the portion of the stream of water where it breaks into drops. If the apparatus is used in an open plane, and care be taken to eliminate all disturbance due to the presence of the electrometer itself and of the observer above the ground, the indicated effect, if expressed in absolute electrostatic measure, and divided by the height of the point tested above the ground, has only to be [according to an old theorem of Coulomb's (see footnote on § 25, above), corrected by Laplace] divided by four times the ratio of the circumference of a circle to its diameter, to reduce it to an expression of the number of units, in absolute electrostatic measure, of the electricity per unit of area of the earth's surface at the time and place. The mathematical theory does away with every difficulty in explaining the various and seemingly irreconcilable views which different writers have expressed, and explanations which different observers have given of the functions of their testing apparatus. In the present state of electric science, the most convenient and generally intelligible way to state the result of an observation of terrestrial atmospheric electricity, in absolute measure, is in terms of the number of elements of a constant galvanic battery, required to produce the same difference of potentials as exists between the earth and a point in the air at a stated height above an open level plane of ground. Observations with the portable electrometer had given, in ordinary fair weather, in the island of Arran, on a flat open sea beach, readings varying from 200 to 400, Daniel's elements, as the difference of potentials between the earth and the match, at a height of 9 feet above it. Hence, the intensity of electric force perpendicular to the earth's surface must have amounted to from 22 to 44 Daniel's elements per foot of air. In fair weather, with breezes from the east or north-east, he had often found from 6 to 10 times the higher of these intensities.

282. Even in fair weather, the intensity of the electric force in the air near the earth's surface is perpetually fluctuating. The speaker had often observed it, especially during calms or very light breezes from the east, varying from 40 Daniel's elements per foot to three or four times that amount during a few minutes; and returning again as rapidly to the lower amount. More frequently he had observed variations from about 30 to about 40, and back again, recurring in uncertain periods of perhaps about two minutes. These gradual variations cannot but be produced by electrified masses of air or cloud, floating by the locality of observation. Again, it is well known that

during storms of rain, hail, or snow, there are great and sometimes sudden variations of electric force in the air close to the earth. These are undoubtedly produced, partly as those of fair weather, by motions of electrified masses of air and cloud; partly by the fall of vitreously or resinously electrified rain, leaving a corresponding deficiency in the air or cloud from which it falls; and partly by disruptive discharges (flashes of lightning) between masses of air or cloud, or between either and the earth. The consideration of these various phenomena suggested the following questions, and modes of observation for answering them:—

283. *Question* 1. How is electricity distributed through the different strata of the atmosphere to a height of five or six miles above the earth's surface in ordinary fair weather? To be answered by electrical observations in balloons at all heights up to the highest limit, and simultaneous observations at the earth's surface.

Q. 2. Does electrification of air close to the earth's surface, or within a few hundred feet of it, sensibly influence the observed electric force? and if so, how does it vary with the weather, and with the time of day or year? The first part of this question has been answered very decidedly in the affirmative, first, for large masses of air within a few hundred yards of the earth's surface, by means of observations made simultaneously at a station near the seashore in the island of Arran, and at one or other of several stations at different distances, within six miles of it, on the sides and summit of Goatfell. After that it was found, by simultaneous observations made at a window in the Natural Philosophy Lecture-Room, and on the College Tower of the University of Glasgow, that the influence of the air within 100 feet of the earth's surface was always sensible at both stations, and often paramount at the lower. Thus, for example, when, in broken weather, the superficial electrification of the outside of the lecture-room, about 20 feet above the ground, in a quadrangle of buildings, was found positive, the superficial electrification of the sides of the tower, about 70 feet higher, was often found negative, or nearly zero; and this sometimes even when the positive electrification of the sides of the building at the lower station equalled in amount an ordinary fair weather negative. This state of things could only exist in virtue of a negative electrification of the circumambient air, inducing a positive electrification on the ground and sides of the quadrangle, but not sufficient to counterbalance the influence, on the higher parts of the tower, of more distant positively electrified aërial masses.

A long continuation of such systems of simultaneous observation—not in a town only, but in various situations of flat and of mountainous country, on the sea coast as well as far inland, in various regions of the world—will be required to obtain the information asked for in the second part of this question.

Q. 3. Do the particles of rain, hail, and snow in falling through the air possess absolute charges of electricity? and if so, whether positive or negative, and of what amounts in different conditions as to place and weather? Attempts to answer this question have been made by various observers, but as yet without success; as, for instance, by an "electro-pluviometer," tried at Kew many years ago. By using a sufficiently well-insulated vessel to collect the falling particles, it is quite certain that a decided answer may be obtained with ease for the cases of hail and snow. Inductive effects produced by drops splashing away from the collecting vessel, if exposed to the electric force of the air in an open position, or inductive effects of the opposite kind produced by drops splashing away from surrounding walls or screens and falling into the collecting vessel, if not in an exposed position, make it less easy to ascertain the electrical quality of rain; but, by taking means to obviate the disturbing effects of these influences, the speaker hoped to arrive at definite results.

284. It would have been more satisfactory to have been able to conclude a discourse on atmospheric electricity otherwise than in questions, but no other form of conclusion would have been at all consistent with the present state of knowledge.

285. The discourse was illustrated by the use of the mirror electrometer reflecting a beam of light from the electric lamp, and throwing it on a white screen, where its motions were measured by a divided scale. The principle of the water-dropping collector was illustrated by allowing a jet of water to flow by a fine nozzle into the middle of the lecture-room, from

an uninsulated metal vessel of water and compressed air, and collecting the drops in an insulated vessel on the floor. This vessel was connected with the testing electrode of the reflecting electrometer; and it was then found to experience a continually increasing negative electrification, when fixed positively electrified bodies were in the neighbourhood of the nozzle. If the same experiment were made in ordinary fair weather in the open air, instead of under the roof and within the walls of the lecture-room, the same result would be observed, without the presence of any artificially electrified body. The vessel from which the water was discharged was next insulated; and other circumstances remaining unvaried, it was shown that this vessel became rapidly electrified to a certain degree of positive potential, and the falling drops ceased to communicate any more electricity to the vessel in which they were gathered.

286. The influence of electrified masses of air was illustrated by carrying about the portable electrometer, with its match burning, to different parts of the lecture-room, while insulated spirit-lamps connected with the positive and negative conductor of an electrical machine, burned on the two sides. The speaker observed the indications on the portable electrometer; but the potentials thus measured were seen by the audience marked on the scale by the spot of light; the reflecting electrometer being kept connected with the portable electrometer in all its positions, by means of a long fine wire. It was found that, when the burning match was on one side of a certain surface dividing the air of the lecture-room, the potential indicated was positive, and on the other side negative.

287. The water-dropping collector constructed for the self-registering apparatus to be used at Kew, had been previously set upon the roof of the Royal Institution, and an insulated wire (Beccaria's "Deferent Wire") led down to the reflecting electrometer on the lecture-room table. The electric force in the air above the roof was thus tested several times during the meeting; and it was at first found to be, as it had been during several days preceding, somewhat feeble positive (corresponding to a feeble negative electrification of the earth's surface, or rather housetops, in the neighbourhood). This was a not unfrequent electrical condition of days, such as these had been

of dull rain, with occasional intervals of heavier rain and of cessation. The natural electricity was again observed by means of the reflecting electrometer during several minutes near the end of the discourse; and was found, instead of the weak positive which had been previously observed, to be strong positive of three or four times the amount. Upon this the speaker quoted[*] an answer which Prior Ceca had given to a question Beccaria had put to him "concerning the state of "electricity when the weather clears up." "'If, when the rain "'has ceased (the Prior said to me) a strong excessive[†] elec- "'tricity obtains, it is a sign that the weather will continue fair "'for several days; if the electricity is but small, it is a sign "'that such weather will not last so much as that whole day, "'and that it will soon be cloudy again, or even will again "'rain.'" The climate of this country is very different from that of Piedmont, where Beccaria and his friend made their observations, but their rule as to the "electricity of clearing weather" has been found frequently confirmed by the speaker. He therefore considered that, although it was still raining at the commencement of the meeting, the electrical indications they had seen gave fair promise[‡] for the remainder of this evening, if not for a longer period. There can be no doubt but that electric indications, when sufficiently studied, will be found important additions to our means for prognosticating the weather; and the speaker hoped soon to see the atmospheric electrometer generally adopted as a useful and convenient weather-glass.

288. The speaker could not conclude without guarding himself against any imputation of having assumed the existence of two electric fluids or substances, because he had frequently spoken of the vitreous and resinous electricities. Dufay's very important discovery of two modes or qualities of electrification, led his followers too readily to admit his supposition of two distinct electric fluids. Franklin, Æpinus, and Cavendish,

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[*] From Beccaria's first letter "On Terrestrial Atmospheric Electricity during Serene Weather."—*Garzegna di Mondovi*, May 16, 1775.

[†] *i.e.*, vitreous, or positive.

[‡] At the conclusion of the meeting it was found that the rain had actually ceased. The weather continued fair during the remainder of the night, and three or four of the finest days of the season followed.

with a hypothesis of one electric fluid, opened the way for a juster appreciation of the *unity* of nature in electric phenomena. Beccaria, with his "electric atmospheres," somewhat vaguely struggled to see deeper into the working of electric force, but his views found little acceptance, and scarcely suggested inquiry or even meditation. The eighteenth century made a school of science for itself, in which, for the not unnatural dogma of the earlier schoolmen, "matter cannot act where it is not," was substituted the most fantastic of paradoxes, *contact does not exist.* Boscovich's theory was the consummation of the eighteenth century school of physical science. This strange idea took deep root, and from it grew up a barren tree, exhausting the soil and overshadowing the whole field of molecular investigation, on which so much unavailing labour was spent by the great mathematicians of the early part of our nineteenth century. If Boscovich's theory no longer cumbers the ground, it is because one true philosopher required more light for tracing lines of electric force.

289. Mr. Faraday's investigation of electrostatic induction influences now every department of physical speculation, and constitutes an era in science. If we can no longer regard electric and magnetic fluids attracting or repelling at a distance as realities, we may now also contemplate as a thing of the past that belief in atoms and in vacuum, against which Leibnitz so earnestly contended in his memorable correspondence with Dr. Samuel Clarke.

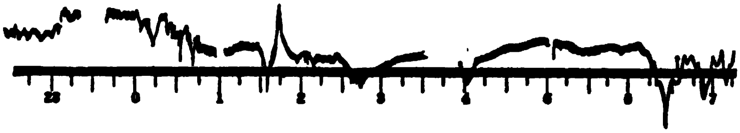
290. We now look on space as full. We know that light is propagated like sound through pressure and motion. We know that there is no substance of caloric—that inscrutably minute motions cause the expansion which the thermometer marks, and stimulate our sensation of heat—that fire is not laid up in coal more than in this Leyden phial, or this weight: there is potential fire in each. If electric force depends on a residual *surface action*, a resultant of an inner tension experienced by the insulating medium, we can conceive that electricity itself is to be understood as not an accident, but an essence of matter. Whatever electricity is, it seems quite certain that electricity in motion IS *heat;* and that a certain alignment of axes of revolution in this motion IS *magnetism.* Faraday's magneto-

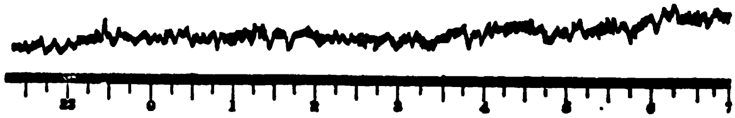
optic experiment makes this not a hypothesis, but a demonstrated conclusion.* Thus a rifle-bullet keeps its point foremost; Foucault's gyroscope finds the earth's axis of palpable rotation; and the magnetic needle shows that more subtle rotatory movement in matter of the earth, which we call terrestrial magnetism: all by one and the same dynamical action.

291. It is often asked, are we to fall back on facts and phenomena, and give up all idea of penetrating that mystery which hangs round the ultimate nature of matter? This is a question that must be answered by the metaphysician, and it does not belong to the domain of Natural Philosophy. But it does seem that the marvellous train of discovery, unparalleled in the history of experimental science, which the last years of the world has seen to emanate from experiments within these walls, must lead to a stage of knowledge, in which laws of inorganic nature will be understood in this sense—that one will be known as essentially connected with all, and in which unity of plan through an inexhaustibly varied execution, will be recognised as a universally manifested result of creative wisdom.

292. [Postscript, with diagram, communicated to the *Philosophical Magazine* in 1861; but now first published.]

Mr. Balfour Stewart, Director of the Kew Meteorological Observatory, has, since the commencement of the present year (1861), brought into regular and satisfactory operation the self-recording atmospheric electrometer with water-dropping collector, described in the preceding abstract: a specimen of the results is exhibited in the accompanying photographic curves.





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[*] See "Dynamical Illustrations of the Magnetic and the Heliçoidal Rotatory Effects of Transparent Bodies on Polarized Light." By Prof. W. Thomson.—*Proceedings of the Royal Society*, June 12, 1856.

293. The diagram exhibits the variations of the electric force of the atmosphere, as photographically recorded by the divided ring electrometer at the Kew Observatory for two successive days, commencing on the 28th of April 1861. The prepared sensitive paper was made to move vertically at a uniform rate by means of clock-work, while a spot of light (the image of a portion of a gas-flame reflected from the mirror of the divided ring electrometer) moved horizontally across it according to the continually varying electric force of the atmosphere, and marked the curve photographically. The datum line, showing the position the spot of light would have if the electric force were zero, is produced by an image from the same source of light reflected from a fixed mirror attached to the case of the electrometer. The numbers indicate hours reckoned from noon as zero, up to 23. The same paper is, for the sake of economy, generally used to bear the record for two days.

Thus the distance of the spot of light from the datum line, on one side or other, indicates, and the photo-chemical action records, for each instant of time the electric potential, positive or negative, of the atmosphere at the point where the stream of water discharged from the insulated vessel breaks into drops.

ON ELECTRICAL "FREQUENCY."

[From *Report of British Association*, Aberdeen Meeting, 1859.]

294. Beccaria found that a conductor insulated in the open air becomes charged sometimes with greater and sometimes with less rapidity, and he gave the name of "frequency" to express the atmospheric quality on which the rapidity of charging depends. It might seem natural to attribute this quality to electrification of the air itself round the conductor, or to electrified particles in the air impinging upon it; but the author gave reasons for believing that the observed effects are entirely due to particles flying away from the surface of the conductor, in consequence of the impact of *non-electrified* particles against it. He had shown in a previous communication that when no electricity of separation (or, as it is more generally called, "frictional electricity," or "contact electricity") is called into

play, the tendency of particles continually flying off from a conductor is to destroy all electrification at the part of its surface from which they break away. Hence a conductor insulated in the open air, and exposed to mist or rain, with wind, will tend rapidly to the same electric potential as that of the air, beside that part of its surface from which there is the most frequent dropping, or flying away, of aqueous particles. The *rapid charging* indicated by the electrometer under cover, after putting it for an instant in connexion with the earth, is therefore, in reality, due to a *rapid discharging* of the exposed parts of the conductor. The author had been led to these views by remarking the extreme rapidity with which an electrometer, connected by a fine wire with a conductor insulated above the roof of his temporary electric observatory in the island of Arran, became charged, reaching its full indication in a few seconds, and sometimes in a fraction of a second, after being touched by the hand, during a gale of wind and rain. The conductor, a vertical cylinder about 10 inches long and 4 inches diameter, with its upper end flat and corner slightly rounded off, stood only 8 feet above the roof, or, in all, 20 feet above the ground, and was nearly surrounded by buildings rising to a higher level. Even with so moderate an exposure as this, sparks were frequently produced between an insulated and an uninsulated piece of metal, which may have been about $\frac{1}{20}$th of an inch apart, within the electrometer, and more than once a continuous line of fire was observed in the instrument during nearly a minute at a time, while rain was falling in torrents outside.

ON THE NECESSITY FOR INCESSANT RECORDING, AND FOR SIMULTANEOUS OBSERVATIONS IN DIFFERENT LOCALITIES, TO INVESTIGATE ATMOSPHERIC ELECTRICITY.

[From *Report of British Association*, Aberdeen Meeting, 1859.]

295. The necessity for incessantly recording the electric condition of the atmosphere was illustrated by reference to observations recently made by the author in the island of Arran, by which it appeared that even under a cloudless sky, without any

sensible wind, the negative electrification of the surface of the earth, always found during serene weather, is constantly varying in degree. He had found it impossible, at any time, to leave the electrometer without losing remarkable features of the phenomenon. Beccaria, Professor of Natural Philosophy in the University of Turin a century ago, used to retire to Garzegna when his vacation commenced, and to make incessant observations on atmospheric electricity, night and day, sleeping in the room with his electrometer in a lofty position, from which he could watch the sky all round, limited by the Alpine range on one side, and the great plain of Piedmont on the other. Unless relays of observers can be got to follow his example, and to take advantage of the more accurate instruments supplied by advanced electric science, a self-recording apparatus must be applied to provide the data required for obtaining knowledge in this most interesting field of nature. The author pointed out certain simple and easily-executed modifications of working electrometers (exhibited to the meeting), to render them self-recording. He also explained a new collecting apparatus for atmospheric electricity, consisting of an insulated vessel of water, discharging its contents in a fine stream from a pointed tube. This stream carries away electricity as long as any exists on its surface, where it breaks into drops. The immediate object of this arrangement is to maintain the whole insulated conductor, including the portion of the electrometer connected with it and the connecting wire, in the condition of no absolute charge; that is to say, with as much positive electricity on one side of a neutral line as of negative on the other. Hence the position of the discharging nozzle must be such, that the point where the stream breaks into drops is in what would be the neutral line of the conductor, if first perfectly discharged under temporary cover, and then exposed in its permanent open position, in which it will become inductively electrified by the aërial electromotive force. If the insulation is maintained in perfection, the dropping will not be called on for any electrical effect, and sudden or slow atmospheric changes will all instantaneously and perfectly induce their corresponding variations in the conductor, and give their appropriate indications to the electrometer. The necessary imperfection of the actual insulation, which tends to bring the neutral line downwards or inwards, or the contrary effects of aërial convection, which, when the insulation is good, generally preponderate, and which in some conditions of the atmosphere, especially during heavy wind and rain, are often very large, are corrected by the tendency of the dropping to maintain the neutral line in the one definite position. The objects to be attained by simultaneous observations in different localities alluded to were—(1.) to fix the constant for any observatory, by which its observations are reduced to absolute measure of electromotive force per foot of air; (2.) to investigate the distribution of electricity in the air itself (whether on visible clouds or in clear air) by a species of electrical trigonometry, of which the general principles were slightly indicated. A portable electrometer, adapted for balloon and mountain observations, with a burning match, regulated by a spring so as to give a cone of fire in the open air, in a definite position with reference to the instrument, was exhibited. It is easily carried, with or without the aid of a shoulder-strap, and can be used by the observer standing up, and simply holding the entire apparatus in his hands, without a stand or rest of any kind. Its indications distinguish positive from negative, and are reducible to absolute measure on the spot. The author gave the result of a determination which he had made, with the assistance of Mr. Joule, on the Links, a piece of level ground near the sea, beside the city of Aberdeen, about 8 A.M. on the preceding day (September 14), under a cloudless sky, and with a light north-west wind blowing, with the insulating stand of the collecting part of the apparatus buried in the ground, and the electrometer removed to a distance of 5 or 6 yards, and connected by a fine wire with the collecting conductor. The height of the match was 3 feet above the ground, and the observer at the electrometer lay on the ground to render the electrical influence of his own body on the match insensible. The result showed a difference of potentials between the earth (negative) and the air (positive) at the match equal to that of 115 elements of Daniell's battery, and, therefore, at that time and place, the aërial electromotive force per foot amounted to that of thirty-eight Daniell's cells, or 1·2 cells per centimetre.

OBSERVATIONS ON ATMOSPHERIC ELECTRICITY.

[From the *Proceedings of the Literary and Philosophical Society of Manchester*, March 1862.]

296. I find that atmospheric electricity is generally negative within doors, and almost always sensible to my divided ring reflecting electrometer. I use a spirit-lamp, on an insulated stand a few feet from walls, floor, or ceiling of my lecture room, and connect it by a fine wire with the insulated half ring of the electrometer. A decided negative effect is generally found, which shows a potential to be produced in the conductors connected with the flame, negative relatively to the earth by a difference amounting to several times the difference of potentials (or electromotive force) between two wires of one metal connected with the two plates of a single element of Daniell's. I have tested that the spirit-lamp gives no *idio-electric* effect amounting to so much as the effect of a single cell. The electric effect observed is therefore not due to thermal or chemical action in the flame. It cannot be due to contact electrifications of metallic or other bodies in conductive communication with the walls, floor, or ceiling, because the potentials of such must always fall short of the difference of potentials produced by a single cell. I have taken care to distinguish the observed natural effect from anything that can be produced by electrical operations for lecture or laboratory purposes. Thus I observe generally in the morning before any electrical operations have been performed, and find ordinarily results quite similar to those observed on the Monday mornings when the electrical machine has not been turned since the previous Friday. The effect, when there has been no artificial disturbance, has *always been found negative, except two or three times*, since the middle of November; but trustworthy observations have not been made on more than a quarter of the number of days.

297. A few turns of the electrical machine, with a spirit-lamp on its prime conductor, or a slightly charged Leyden phial, with its inside coating positive put in connexion with an insulated spirit-lamp, is enough to reverse the common negative indication. Another very striking way in which this may be done is to put a *negatively charged Leyden phial* below an insulated

flame (a common gas-burner, for instance). The flame, becoming positively electrified by induction, keeps throwing off, by the dynamic power of its burning, portions of its own gaseous matter, and does not allow them to be electrically attracted down to the Leyden phial, but forces them to rise. These, on cooling, become, like common air, excellent non-conductors,* and, mixing with the air of the room, give a preponderance of positive influence to the testing insulated flame (that is to say, render the air potential positive at the place occupied by this flame).

298. Half an hour, or often much more, elapses after such an operation, before the natural negatively electrified air becomes again paramount in its influence on the testing flame.

299. That either positive or negative electricity may be carried, even through narrow passages, by air, I have tested by turning an electric machine, with a spirit-lamp on its prime conductor, for a short time in a room separated from the lecture room by an oblique passage about two yards long and then stopping the machine and extinguishing the lamp; so as to send a limited quantity of positive electricity into the air of that room. When the lecture-room window was kept open, and the door leading to the adjoining room shut, the testing spirit-lamp showed the natural negative. When the window was closed, and a small chink (an inch or less wide) opened of the door, the indication quickly became positive. If the door was then shut, and the window again opened, the natural effect was slowly recovered. A current of air, to feed the lecture-room fire, was found entering by either door or window when the other was shut. This alternate positive and negative electric ventilation may be repeated many times without renewing the positive electricity of the adjoining room by turning the machine afresh.

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* I find that steam from a kettle boiling briskly on a common fire is an excellent insulator. I allow it to blow for a quarter of an hour or more against an insulated electrified conductor, without discovering that it has any effect on the retention of the charge. The electricity of the steam itself, in such circumstances, as is to be expected from Faraday's investigation, is not considerable. Common air loses nearly all its resisting power at some temperature between that of boiling water and red-hot iron, and conducts continuously (not, as I believe is generally supposed to be the case, by disruption) as glass does at some temperature below the boiling point, with so great ease as to discharge any common insulated conductor almost completely in a few seconds.

300. The out of doors air potential, as tested by a portable electrometer in an open place, or even by a water dropping nozzle outside, two or three feet from the walls of the lecture room, was generally on these occasions positive, and the earth's surface itself, therefore, of course, negative;—the common fair weather condition, which I am forced to conclude is due to a paramount influence of positive electricity in higher regions of the air, notwithstanding the negative electricity of the air in the lower stratum near the earth's surface. On the two or three occasions when the in-door atmospheric electricity was found positive, and, therefore, the surface of the floor, walls, and ceiling negative, the potential outside was certainly positive, and the earth's surface out of doors negative, as usual in fair weather.

ON SOME REMARKABLE EFFECTS OF LIGHTNING OBSERVED IN A FARM-HOUSE NEAR MONIEMAIL, CUPAR-FIFE.

(From *Proceedings of the Philosophical Society of Glasgow.*)

301. The following is an extract from a letter, addressed last autumn to me by Mr. Leitch, minister of Moniemail parish:—

"MONIEMAIL MANSE, CUPAR-FIFE,
26th August 1849.

" . . . We were visited on the 11th inst. with a violent thunder-storm, which did considerable damage to a farm-house in my immediate neighbourhood. I called shortly afterwards and brought away the wires and the paper which I enclose. . . .

"I have some difficulty in accounting for the appearance of the wires. You will observe that they have been partially fused, and when I got them first they adhered closely to one another. You will find that the flat sides exactly fit. They were both attached to one crank, and ran parallel to one another. The question is, how were they attracted so powerfully as to be compressed together? . . .

"You will observe that the paper is discoloured. This has been done, not by scorching, but by having some substance deposited on it. There was painted *wood* also discoloured, on which the stratum was much thicker. It could easily be rubbed off, when you saw the paint quite fresh beneath. . . .

"The farmer showed me a probang which hung on a nail. The handle only was left. The rest, consisting of a twisted cane, had entirely disappeared. By minute examination I found a small fragment, which was not burnt, but broken off."

[The copper wires and the stained paper, enclosed with Mr. Leitch's letter, were laid before the Society.]

The remarkable effects of lightning, described by Mr. Leitch, are all extremely interesting. Those with reference to the copper wires are quite out of the common class of electrical phenomena; nothing of the kind having, so far as I am aware, been observed previously, either as resulting from natural discharges, or in experiments on electricity. It is not improbable that they are due to the electro-magnetic attraction which must have subsisted between the two wires during the discharge, it being a well-known fact that adjacent wires, with currents of electricity in similar directions along them, attract one another. It may certainly be doubted whether the inappreciably short time occupied by the electrical discharge could have been sufficient to allow the wires, after having been drawn into contact, to be pressed with sufficient force to make them adhere together, and to produce the remarkable impressions which they still retain. On the other hand, the electro-magnetic force must have been very considerable, since the currents in the wires were strong enough nearly to melt them, and since they appear to have been softened, if not partially fused; the flattening and remarkable impressions might readily have been produced by even a slight force subsisting after the wires came in contact.

The circumstances with reference to the probang, described by Mr. Leitch, afford a remarkable illustration of the well-known fact, that an electrical discharge, when effected through the substance of a non-conducting (that is to say, a *powerfully resisting*) solid, shatters it, without producing any considerable elevation of its temperature; not leaving marks of combustion, if it be of an ordinary combustible material such as wood.

Dr. Robert Thomson, at my request, kindly undertook to examine the paper removed from the wall of the farm-house, and enclosed with his letter to me by Mr. Leitch; so as, if possible, by the application of chemical tests, to discover the staining substance deposited on its surface. Mr. Leitch, in his letter, had suggested that it would be worth while to try whether this case is an example of the deposition of sulphur, which Fusinieri believed he had discovered in similar circumstances. Accordingly tests for sulphur were applied, but with entirely negative results. Stains presenting a similar appearance had been sometimes observed on paper in the neighbourhood of copper-wires through which powerful discharges in experiments with the hydro-electric machine had been passed; and from this it was suggested that the staining substance might have come from the bell wires. Tests for copper were accordingly applied, and the results were most satisfactory. The front of the paper was scraped in different places, so as to remove some of the pigment in powder; and the powders from the stained, and from the not stained parts, were repeatedly examined. The presence of copper in the former was readily made manifest by the ordinary tests: in the latter, no traces of copper could be discovered. The back of the paper presented a green tint, having been torn from a wall which has probably been painted with Scheele's green; and matter scraped away from any part of the back was found to contain copper. Since, however, the stains in front were manifestly superficial, the discolouration being entirely removed by scraping, and since there was no appearance whatever of staining at the back of the paper, nor of any effect of the electrical discharge, it was impossible to attribute the stains to copper produced from the Scheele's green on the wall below the paper. Dr. Thomson, therefore, considered the most probable explanation to be, that the stains of oxide of copper must have come from the bell-wire. To ascertain how far this explanation could be supported by the circumstances of the case, I wrote to Mr. Leitch asking him for further particulars, especially with reference to this point, and I received the following answer:—

"MONIMAIL, CUPAR-FIFE,
30th Nov. 1849.

". . . I received your letter to-day, and immediately called at Hall-hill, in the parish of Collessie, the farm-house which had been struck by the lightning. . . .

"I find that Dr. Thomson's suggestion is fully borne out by the facts. I at first thought that the bell-wire did not run along the line of discolouration, but I now find that such was the case. . . .

[From a drawing and explanation which Mr. Leitch gives, it appears that the wire runs vertically along a corner of the room, from the floor, to about a yard from the ceiling, where it branches into two, connected with two cranks near one another, and close to the ceiling.]

"The efflorescence [the stains previously adverted to] was on each side of this perpendicular wire. In some places it extended more than a foot from the wire. The deposit seemed to vary in thickness according to the surface on which it was deposited. There was none on the plaster on the roof. It was thinnest upon the wall-paper, and thickest upon the wood facing of the door.* This last exhibited various colours. On the thickest part it appeared quite black; where there was only a slight film, it was green or yellow. . . .

"I may mention that the thunder-storm was that of the 11th of August last. It passed over most of Scotland, and has rarely been surpassed for terrific grandeur at least beyond the tropics. It commenced about nine o'clock P.M., and in the course of an hour it seemed to die away altogether. The peals became very faint, and the intervals between the flashes and the reports very great, when all at once a terrific crashing peal was heard, which did the damage. The storm ceased with this peal.

"The electricity must have been conducted along the lead on the ridge of the house, and have diverged into three streams; one down through the roof, and the two others along the roof to the chimneys. One of these appears to have struck a large stone out from the chimney, and to have been conducted down the chimney to the kitchen, where it left traces upon the floor. It had been washed over before I saw it, but still the traces were visible on the Arbroath flags. The stains were of a lighter tint than the stone, and the general appearance was as if a pail of some light-coloured fluid had been dashed over the floor, so as to produce various distinct streams. All along the course of the discharge, and particularly in the neighbourhood of the bell-wires, there were small holes in the wall about an inch deep, like the marks that might be made by a finger in soft plaster.

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* These remarkable facts are probably connected with the conducting powers of the different surfaces. The plaster on the roof is not so good a conductor as the wall-paper, with its pigments; and the painted wood is probably a better conductor than either.—W. T.

"Most of the windows were shattered, and all the fragments of glass were on the outside. I suppose this must be accounted for by the expansion of the air within the house.

"The window-blind of the staircase, which was down at the time, was riddled, as if with small shot. The diameter of the space so riddled was about a foot. On minute examination I found that the holes were not such as could readily be made by a pointed instrument or a pellet. They were angular, the cloth being torn along both the warp and the woof.

"The house was shattered from top to bottom. Two of the serving-maids received a positive shock, but soon recovered. A strong smell of what was supposed to be sulphur was perceived throughout the house, but particularly in the bed-room in which the effects I described before took place."

XVII. SOUND PRODUCED BY THE DISCHARGE OF A CONDENSER.

[LETTER TO PROFESSOR TAIT.]

KILMICHAEL, BRODICK,
ISLE OF ARRAN, *Oct.* 10, 1863.

302. Yesterday evening, when engaged in measuring the electrostatic capacities of some specimens of insulated wire designed for submarine telegraph cables, I had occasion frequently to discharge, through a galvanometer coil, a condenser consisting of two parallel plates of metal, separated by a space of air about ·007 inch across, and charged to a difference of potentials equal to that of about 800 Daniell's elements. I remarked at an instant of discharge a sharp sound, with a

very slight prolonged resonance, which seemed to come from the interior of the case containing the condenser, and which struck me as resembling a sound I had repeatedly heard before when the condenser had been overcharged and a spark passed across its air-space. But I ascertained that this sound was distinctly audible when there was no spark within the condenser, and the whole discharge took place fairly through the 2000 yards of fine wire, constituting the galvanometer coil. I arranged the circuit so that the place where the contact was made to produce the discharge was so far from my ear that the initiating spark was inaudible; but still I heard distinctly the same sound as before from within the condenser.

303. Using instead of the galvanometer coil either a short wire or my own body (as in taking a shock from a Leyden phial), I still heard the sound within the condenser. The shock was imperceptible except by a very faint prick on the finger in the place of the spark, and (the direct sound of the spark being barely, if at all, sensible) there was still a very audible sound, always of the same character, within the condenser, which I heard at the same instant as I felt the spark on my finger. Mr. Macfarlane could hear it distinctly standing at a distance of several yards. We watched for light within the condenser, but could see none. I have since ascertained that suddenly charging the condenser out of one of the specimens of cable charged for the purpose produces the same sound within the condenser; also that it is produced by suddenly reversing the charge of the condenser.

304. Thus it is distinctly proved that a plate of air emits a sound on being suddenly subjected to electric force, or on experiencing a sudden change of electric force through it. This seems a most natural result when viewed in connexion with the new theory put forward by Faraday in his series regarding the part played by air or other dielectric in manifestations of electric force. It also tends to confirm the hypothesis I suggested to account for the remarkable observation made regarding lightning, when you told me of it about a year ago, and other similar observations which I believe have been reported, proving a sound to be heard at the instant of a flash of lightning in localities at considerable distances from any part of the line of discharge, and which by some have been supposed to de-

monstrate an error in the common theory of sound. I may add that Mr. Macfarlane tells me he believes he has heard, at the instant of a flash of lightning, a sound as of a heavy body striking the earth, and imagined at first that something close to him had been struck, but heard the ordinary thunder at a sensible time later.

XVIII. MEASUREMENT OF THE ELECTROSTATIC FORCE PRODUCED BY A DANIELL'S BATTERY.

[*Proceedings Royal Society*, Feb. 23 and April 12, 1860, or *Phil. Mag.* 1860, second half-year.]

305. In a paper "On Transient Electric Currents," published in the *Philosophical Magazine* for June 1853, I described a method for measuring differences of electric potential in absolute electrostatic units, which seemed to me the best adapted for obtaining accurate results. The "absolute electrometer" which I exhibited to the British Association on the occasion of its meeting at Glasgow in 1855, was constructed for the purpose of putting this method into practice, and, as I then explained, was adapted to reduce the indications of an electroscopic* or of a torsion electrometer to absolute measure.

306. The want of sufficiently constant and accurate instruments of the latter class has long delayed my carrying out of the plans then set forth. Efforts which I have made to produce electrometers to fulfil certain conditions of sensibility, convenience, and constancy, for various objects, especially the electrostatic measurement of galvanic forces, and of the differences of potential required to produce sparks in air, under definite conditions, and the observation of natural atmospheric electricity, have enabled me now to make a beginning of absolute determinations, which I hope to be able to carry out soon in a much more accurate manner. In the meantime, I shall give a slight description of the chief instruments and processes

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* I have used the expression "electroscopic electrometer," to designate an electrometer of which the indications are merely read off in each instance by a single observation, without the necessity of applying any experimental process of weighing, or of balancing by torsion, or of otherwise modifying the conditions exhibited.

followed, and state the approximate results already obtained, as these may be made the foundation of various important estimates in several departments of electrical science.

307. The absolute electrometer alluded to above (compare § 358, below), consists of a plane metallic disc, insulated in a horizontal position, with a somewhat smaller plane metallic disc hung centrally over it, from one end of the beam of a balance. A metal case protects the suspended disc from currents of air, and from irregular electric influences, allowing a light vertical rod, rigidly connected with the disc at its lower end, and suspended from the balance above, to move up and down freely, through an aperture just wide enough not to touch it. In the side of the case there is another aperture, through which projects an electrode rigidly connected with the lower insulated disc. The upper disc is kept in metallic communication with the case.

308. In using this instrument to reduce the indications of an electroscopic or torsion electrometer to absolute electrostatic measure, the insulated part of the electrometer is kept in metallic communication with the insulated disc, while the cases enclosing the two instruments are also kept in metallic communication with one another. A charge, either positive or negative, is communicated to the insulated part of the double apparatus. The indication of the tested electrometer is read off, and at the same time the force required to keep the moveable disc at a stated distance from the fixed disc below it, is weighed by the balance. This part of the operation is, as I anticipated, somewhat troublesome, in consequence of the instability of the equilibrium, but with a little care it may be managed with considerable accuracy. The plan which I have hitherto followed, has been to limit the play of the arm of the balance to a very small arc, by means of firm stops suitably placed, thus allowing a range of motion to the upper disc through but a small part of its whole distance from the lower. A certain weight is put into the opposite scale of the balance, and the indications of the second electrometer are observed when the electric force is just sufficient to draw down the upper disc from resting in its upper position, and again when insufficient to keep it down with the beam pressed on its lower stop. This operation is repeated at different distances,

and thus no considerable error depending on a want of parallelism between the discs could remain undetected. It may be remarked that the upper disc is carefully balanced by means of small weights attached to it, so as to make it hang as nearly as possible parallel to the lower disc. The stem carrying it is graduated to hundredths of an inch (·254 of a millimetre); and by watching it through a telescope at a short distance, it is easy to observe $\frac{1}{10}$ of a millimetre of its vertical motion.

309. I have recently applied this method to reduce to absolute electrostatic measure the indications of an electrometer forming part of a portable apparatus for the observation of atmospheric electricity. In this instrument (compare § 263) a very light bar of aluminium attached at right angles to the middle of a fine platinum wire, which is firmly stretched between the inside coatings of two Leyden phials, one occupying an inverted position above the other, experiences and indicates the electrical force which is the subject of measurement, and which consists of repulsions in contrary directions on its two ends, produced by two short bars of metal fixed on the two sides of the top of a metal tube, supported by the inside coating of the lower phial.

310. The amount of the electrical force (or rather, as it should be called in correct mechanical language, *couple*) is measured by the angle through which the upper Leyden phial must be turned round an axis coincident with the line of the wire, so as to bring the index to a marked position. An independently insulated metal case, bearing an electrode projecting outwards, to which the body to be tested is applied, surrounds the index and repelling bars, but leaves free apertures above and below, for the wire to pass through it without touching it; and by other apertures in its sides and top, it allows the motions of the index to be observed, and the Leyden phials to be charged or discharged at pleasure, by means of an electrode applied to one of the fixed bars described above. When by means of such an electrode the inside coatings of the Leyden phials are kept connected with the earth, this electrometer becomes a plain repulsion electrometer, on the same principle as Peltier's, with the exception that the index, supported by a platinum wire instead of on a pivot, is directed by elasticity of torsion instead

of by magnetism; and the electrical effect to be measured is produced by applying the electrified body to a conductor connected with a fixed metal case round the index and repelling bars, instead of with these conductors themselves.

311. This electrometer, being of suitable sensibility for direct comparison with the absolute electrometer according to the process described above, is not sufficiently sensitive to measure directly the electrostatic effect of any galvanic battery of fewer than two hundred cells with much accuracy. Not having at the time arrangements for working with a multiple battery of reliable character, I used a second torsion electrometer of a higher degree of sensibility as a medium for comparison, and determined the value of its indications by direct reference to a Daniell's battery of from six to twelve elements in good working order. This electrometer, in which a light aluminium index, suspended by means of a fine glass fibre, kept constantly electrified by means of a light platinum wire hanging down from it and dipping into some sulphuric acid in the bottom of a charged Leyden jar, exhibits the effects of electric force due to a difference of potentials between two halves of a metallic ring separately insulated in its neighbourhood, will be sufficiently described in another communication to the Royal Society. Slight descriptions of trial instruments of this kind have already been published in the *Transactions of the Pontifical Academy of Rome*,* and in the second edition of Nichol's *Cyclopædia* (article Electricity, Atmospheric), 1860 (§§ 249, 266, above).

312. I hope soon to have another electrometer on the same general principle, but modified from those hitherto made, so as to be more convenient for accurate measurement in terms of constant units. In the meantime, I find that, by exercising sufficient care, I can obtain good measurements by means of the divided ring electrometer of the form described in Nichol's *Cyclopædia* (§ 263, above).

313. In the ordinary use of the portable electrometer, a considerable charge is communicated to the connected inside coatings of the Leyden phials, and the aluminium index is brought to an accurately marked position by torsion, while the insulated

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* Accademia Pontificia dei Nuovi Lyncei, February 1857.

metal case surrounding it is kept connected with the earth. The square root of the reading of the torsion-head thus obtained measures the potential, to which the inside coatings of the phials have been electrified. If, now, the metal case referred to is disconnected from the earth and put in connexion with a conductor whose potential is to be tested, the square root of the altered reading of the torsion-head required to bring the index to its marked position in the new circumstances measures similarly the difference between this last potential and that of the inside coatings of the phials. Hence the excess of the latter square root above the former expresses in degree and in quality (positive or negative) the required potential. This plan has not only the merit of indicating the quality of the electricity to be tested, which is of great importance in atmospheric observation, but it also affords a much higher degree of sensibility than the instrument has when used as a plain repulsion electrometer; and, on account of this last-mentioned advantage, it was adopted in the comparisons with the divided ring electrometer. On the other hand, the portable electrometer was used in its least sensitive state, that is to say, with its Leyden phials connected with the earth, when the comparisons with the absolute electrometer were made.

314. The general result of the weighings hitherto made, is that when the discs of the absolute electrometer were at a distance of ·5080 of a centimetre, the number of degrees of torsion in the portable electrometer was ·20924 times the number of grammes' weight required to balance the attractive force; and the number of degrees of torsion was ·4983 times the number of grammes' weight found in other series of experiments in which the distance between the discs was ·762 of a centimetre. According to the law of inverse squares of the distances to which the attraction between two parallel discs is subject when a constant difference of potentials is maintained between them,[*] the force at a distance of ·254 of a centimetre would have been $\frac{1}{12\cdot48}$, according to the first of the preceding results, or, according to the second, $\frac{1}{13\cdot28}$ of the number of degrees of torsion. The mean of these is $\frac{1}{12\cdot87}$, or ·0777; and we may consider this

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[*] See § 11 of Elements of Mathematical Theory of Electricity appended to the communication following this in the "Proceedings."

number as representing approximately the value in grammes' weight at ·254 of a centimetre distance between the discs of the absolute electrometer, corresponding to one degree of torsion of the portable electrometer. By comparing the indications of the portable electrometer with those of the divided ring electrometer, and by evaluating those of the latter in terms of the electromotive force of a Daniell's battery charged in the usual manner, I find that 284 times the square root of the number of degrees of torsion in the portable electrometer is approximately the number of cells of a Daniell's battery which would produce an electromotive force (or, which is the same thing, a difference of potentials) equal to that indicated. Hence the attraction between the discs of the portable electrometer, if at ·254 of a centimetre distance, and maintained at a difference of potentials amounting to that produced by 284 cells, is ·0777 of a gramme. The effect of 1000 cells would therefore be to give a force of ·965 of a gramme, since the force of attraction is proportional to the square of the difference of potentials between the discs. The diameter of the opposed circular areas between which the attraction observed took place, was 14·88 centimetres. Its area was therefore 174·0 square centimetres, and therefore the amount of attraction per square decimetre, according to the preceding estimate for ·254 of a centimetre distance and 1000 cells' difference of potential, is ·554 of a gramme. Hence, with an electromotive force or difference of potentials produced by 1000 cells of Daniell's battery, the force of attraction would be 3·57 grammes weight per square decimetre between discs separated to a distance of 1 millimetre. [The force in grammes weight is equal to $\cdot 000357 \times n^2$, if the area of each of the opposed surfaces is equal to *a square* whose side is n times the distance between them, provided n be a large number.]

315. This result differs very much from an estimate I have made according to Weber's comparison of electrostatic with electro-magnetic units and my theoretical estimate of 2,500,000 British electro-magnetic units for the electromotive force of a single element of Daniell's. On the other hand, it agrees to a remarkable degree of accuracy with direct observations made for me, during my absence in Germany, by Mr. Macfarlane, in

the months of June and July 1856, on the force of attraction produced by the direct application of a miniature Daniell's battery, of different numbers of elements, from 93 to 451, applied to the same absolute electrometer with its discs at ·2006 of a centimetre asunder. These observations gave forces varying, on the whole, very closely according to the square of the number of cells used; and the mean result reduced according to this law to 1000 cells was 1·516 grammes. Reducing this to the distance of 1 millimetre, and dividing by 1·74, the area in square decimetres, we find 3·51 grammes per square decimetre at a distance of 1 millimetre.

316. Although the experiments leading to this result were executed with great care by Mr. Macfarlane, I delayed publishing it because of the great discrepance it presented from the estimate which I deduced from Weber's measurement, published while my preparations were in progress. I cannot doubt its general correctness now, when it is so decidedly confirmed by the electrometric experiments I have just described, which have been executed chiefly by Mr. John Smith and Mr. John Ferguson, working in my laboratory with much ability since the month of November. I am still unable to explain the discrepance, but it may possibly be owing to some miscalculation I have made in my deductions from Weber's result.

GLASGOW COLLEGE, *Jan.* 18, 1860.

[*Addition, April* 1870.—From experiments of the present date, performed by Mr. William Leitch and Mr. Dugald M'Kichan, with the new Absolute Electrometer (§ 364, below), it is deduced that with the difference of potentials produced by 1000 Daniell's cells in series, the force of attraction would be 5·7 grammes per square decimetre between discs separated to a distance of 1 millimetre, instead of 3·57 grammes as found in § 314. This new measurement, with Maxwell's correction of Weber's number, which diminishes it by about 8 per cent. (*Report of British Association for* 1869, page 438 :—Committee on Electrical Standards), seems to reduce to as nearly as may be nothing, the discrepance from my thermo-dynamic estimate of December 1851 (*Philosophical Magazine*) referred to in § 318, below. Calculating from it by § 339, we find 3·74 for the difference of potentials (or electromotive force in absolute electrostatic measure), produced by 1000 elements of Daniell's.]

POSTSCRIPT, *April* 12, 1860.

317. I have since found that I had inadvertently misinterpreted Weber's statement in the ratio of 2 to 1. I had always, as it appears to me most natural to do, regarded the transference of negative electricity in one direction, and of positive electricity in the other direction, as identical agencies, to which, in our ignorance as to the real nature of electricity, we may apply indiscriminately the one expression or the other, or a combination of the two. Hence I have always regarded a current of unit strength as a current in which the positive or vitreous electricity flows in one direction at the rate of a unit of electricity per unit of time; or the negative or resinous electricity in the other direction at the same rate; or (according to the infinitely improbable hypothesis of two electric fluids) the vitreous electricity flows in one direction at any rate less than a unit per second, and the resinous in the opposite direction at a rate equal to the remainder of the unit per second. I have only recently remarked that Weber's expressions are not only adapted to the hypothesis of two electric fluids, but that they also reckon as a current of unit strength, what I should have called a current of strength 2, namely, a flow of vitreous electricity in one direction at the rate of a unit of vitreous electricity per unit of time, and of the resinous electricity in the other direction simultaneously, at the rate of a unit of resinous electricity per unit of time.

318. Weber's result as to the relation between electrostatic and electro-magnetic units, when correctly interpreted, I now find would be in perfect accordance with my own results given above, if the electromotive force of a single element of the Daniell's battery used were 2,140,000 British electro-magnetic units instead of 2,500,000, as according to my thermo-dynamic estimate. This is as good an agreement as could be expected when the difficulties of the investigations, and the uncertainty which still exists as to the true measure of the

electromotive force of the Daniell's element are considered. It must indeed be remarked that the electromotive force of Daniell's battery varies by two or three or more per cent. with variations of the solutions used; that it varies also very sensibly with temperature; and that it seems also to be dependent, to some extent, on circumstances not hitherto elucidated. A thorough examination of the electromotive force of Daniell's and other forms of galvanic battery, is an object of high importance, which, it is to be hoped, will soon be attained. Until this has been done, at least for Daniell's battery, the results of the preceding paper may be regarded as having about as much accuracy as is desirable.

319. I may state, therefore, in conclusion, that the average electromotive force per cell of the Daniell's batteries which I have used, produces a difference of potentials amounting to ·00296 [corrected to ·00374, April 1870,] in absolute electrostatic measure. This statement is perfectly equivalent to the following in more familiar terms:—

One thousand cells of Daniell's battery, with its two poles connected by wires with two parallel plates of metal 1 millimetre apart, and each a square decimetre in area, produces an electrical attraction equal to the weight of 3·57 [corrected to 5·7] grammes.

XIX. MEASUREMENT OF THE ELECTROMOTIVE FORCE REQUIRED TO PRODUCE A SPARK IN AIR BETWEEN PARALLEL METAL PLATES AT DIFFERENT DISTANCES.

[*Proceedings Royal Society*, Feb. 23 and April 12, 1860, or *Phil. Mag.*, 1860, second half-year.]

320. THE electrometers used in this investigation were the absolute electrometer and the portable electrometer described in my last communication to the Royal Society, and the operations were executed by the same gentlemen, Mr. Smith and Mr. Ferguson. The conductors between which the sparks passed were two unvarnished plates of a condenser, of which one was moved by a micrometer screw, giving a motion of $\frac{1}{25}$ of an inch (about one millimetre) per turn, and having its head divided into 40 equal parts of circumference. The readings on the screw-head could be readily taken to tenth parts of a division, that is to say, to about $\frac{1}{400}$ of a millimetre on the distance to be measured. The point from which the spark would pass in successive trials being somewhat variable, and often near the edges of the discs, a thin flat piece of metal, made very slightly convex on its upper surface like an extremely flat watch-glass, was laid on the lower plate. It was then found that the spark always passed between the crown of this convex piece of metal and the flat upper plate. The curvature of the former was so small, that the physical circumstances of its own electrification near its crown, the opposite electrification of the opposed flat surface in the parts near the crown of the convex, and the electric pressure on or tension in the air between them could not, it was supposed, differ sensibly from those between two plane conducting surfaces at the same distance and maintained at the same difference of potentials.

321. The reading of the screw-head corresponding to the position of the moveable disc when touching the metal below, was always determined electrically by making a succession of sparks pass, and approaching the moveable disc gradually by the screw until all appearance of sparks ceased. Contact was thus produced without any force of pressure between the two bodies capable of sensibly distorting their supports.

With these arrangements several series of experiments were made, in which the differences of potentials producing sparks across different thicknesses of air were measured first by the absolute electrometer, and afterwards by the portable torsion electrometer. The following Tables exhibit the results hitherto obtained:—

322. TABLE I.—*December 13, 1859. Measurements by absolute electrometer of maximum electrostatic forces* across a stratum of air of different thicknesses.*

Area of each plate of absolute electrometer = 174 square centimetres.
Distance between plates of absolute electrometer = ·508 of a centimetre.

Length of spark in inches. s.	Weight in grains required to balance in absolute electrometer. w.	Electromotive force in units of the electrometer. $\sqrt{w}$.	Electrostatic force, or electromotive force per inch of air, in temporary units. $\frac{\sqrt{w}}{s}$.
·007	6	2·4495	349·9
·0105	9	3·0000	285·7
·0115	10	3·1622	275·0
·014	13	3·6055	257·5
·017	16	4·0000	235·3
·018	19	4·3589	242·2
·024	30	5·4772	228·2
·0295	40	6·3245	214·4
·034	50	7·0710	208·0
·0385	60	7·7459	201·2
·041	70	8·3666	204·1
·0445	80	8·9442	201·0
·048	90	9·4868	197·6
·052	100	10·0000	192·3
·055	110	10·4880	190·7
·058	120	10·9544	188·9
·060	130	11·4017	190·0

323. These numbers demonstrate an unexpected and a very remarkable result,—that greater electromotive force per unit

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* See § 331 below.

length of air is required to produce a spark at short distances than at long. When it is considered that the absolute electrification of each of the opposed surfaces* depends simply on the electromotive force per unit length of the space between them, or, which is the same thing, the resultant electrostatic force in the air occupying that space, it is difficult even to conjecture an explanation. Without attempting to explain it, we are forced to recognise the fact that a thin stratum of air is stronger than a thick one against the same disruptive tension in the air, according to Faraday's view of its condition as transmitting electric force, or against the same lifting electric pressure from its bounding surfaces, according to the views of the eighteenth century school, as represented by Poisson. The same conclusion is established by a series of experiments with the previously-described portable torsion electrometer substituted for the absolute electrometer, leading to results shown in the following Table :—

324. TABLE II.—*January* 17, 1860. ***Measurements by portable torsion electrometer of electromotive forces producing sparks across a stratum of air of different thicknesses.***

Length of spark in inches. s.	Torsion in degrees required to balance in electrometer. θ.	Electromotive force in units of the electrometer. $\sqrt{\theta}$.	Electrostatic force, or electromotive force per inch of air, in temporary units. $\sqrt{\theta} \div s$.
·001	3	1·732	1732
·002	7	2·646	1323
·003	11	3·316	1105
·004	14	3·742	935
·005	18	4·243	849
·006	22	4·690	782
·007	27	5·196	742
·008	30	5·477	685
·009	33	5·744	638
·010	38	6·164	616
·011	43	6·557	596
·012	48·5	6·964	580
·013	54	7·348	565
·014	59	7·681	549
·015	66	8·124	542
·016	73	8·544	534
·017	79	8·888	523
·018	85	9·219	512

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* See § 332 below.

325. The series of experiments here tabulated stops at the distance 18 thousandths of an inch, because it was found that the force in the electrometer corresponding to longer sparks than that, was too strong to be measured with certainty by the portable electrometer, whether from the elasticity of the platinum wire, or from the rigidity of its connexion with the aluminium index being liable to fail when more than 85° or 90° of torsion were applied. So far as it goes, it agrees remarkably well with the other experiments exhibited in Table I., as is shown by the following comparative Table, in which, along with results of actual observation extracted from Table II., are placed results deduced from Table I. by interpolation for the same lengths of spark:—

TABLE III.—*Experiments of December* 13, 1859, *and January* 17, 1860, *compared.*

Col. 1. Length of spark in inches. s.	Col. 2. Electromotive force per inch of air, Dec. 13, in temporary units of that day. $\frac{\sqrt{w}}{s}$.	Col. 3. Electromotive force per inch of air, Jan. 17, in temporary units of that day. $\frac{\sqrt{\theta}}{s}$.	Col. 4. Ratios of numbers in Col. 3 to numbers in Col. 2.
·007	349·3	742	2·13
·0105	285·7	606	2·12
·0115	275·0	588	2·14
·014	257·5	549	2·14
·017	235·3	523	2·22
·018	242·2	512	2·11
			Mean 2·14

The close agreement with one another of the numbers in Col. 4, derived from series differing so much as those in Cols. 2 and 3, and obtained by means of electrometers differing so much in construction, constitutes a very thorough confirmation of the remarkable result inferred above from the experiments of the first series, and shows that the law of variation of the electrostatic force in the air required to produce sparks of the different lengths, must be represented with some degree of accuracy by the numbers shown in the last column of either Table I. or Table III.

The following additional series of experiments were made on precisely the same plan as those of Table II.:—

TABLE IV.—*January* 21, 1860. *Measurements by portable torsion electrometer of electromotive forces producing sparks across a stratum of air of different thicknesses.*

Length of spark in inches. s.	Torsion in degrees required to balance, in electrometer. θ.	Electromotive force in units of the electrometer. $\sqrt{\theta}$.	Electrostatic force, or electromotive force per inch of air, in temporary units. $\sqrt{\theta} \div s$.
·001	3·2	1·79	1790
·002	6·4	2·32	1160
·003	10·5	3·24	1080
·004	13·2	3·63	907
·005	14·2	3·77	754
·006	18·2	4·27	712
·007	21·7	4·66	666
·012	41·2	6·42	535
·013	46·7	6·83	525
·014	53·2	7·29	521
·015	57·2	7·56	504
·016	63·2	7·95	497
·017	68·2	8·26	486
·018	78·2	8·84	491

TABLE V.—*January* 23, 1860. *Similar experiments repeated.*

s.	θ.	$\sqrt{\theta}$.	$\sqrt{\theta} \div s$.
·001	3·5	1·87	1870
·002	6·5	2·55	1275
·003	9·5	3·08	1027
·004	12·7	3·56	890
·005	15·5	3·94	788
·006	18·5	4·30	716
·007	23·0	4·80	686
·008	25·62	5·06	632
·009	30·5	5·52	613
·010	35·0	5·92	592
·011	39·5	6·28	571
·012	44·0	6·63	553
·013	50·0	7·07	544
·014	54·0	7·35	525
·015	59·0	7·68	512
·016	63·5	7·97	498
·017	69·5	8·34	490
·018	74·5	8·63	479

The difference between the numbers shown in these two Tables and in Table II. above, are probably due in part to true differences in the resistance of the air to electrical disruption; but variations in the electrometer, which was by no means of perfect construction, may have sensibly influenced the results, especially as regards the differences between those shown in Table II. and those shown in Tables IV. and V., which, agreeing on the whole closely with one another, fall considerably short of the former.

326. Table VI.—*Summary of results reduced to absolute measure.*

Col. 1. Length of spark in centimetres. s.	Col. 2. Electrostatic forces according to simple determinations of Dec. 13, 1859. $\frac{\sqrt{w}}{s}\times\cdot508\sqrt{\frac{981\cdot4\times8\pi^*}{174}}=R$.	Col. 3. Electrostatic forces according to estimated average of various determinations. R.	Col. 4. Differences.	Col. 5. Pressures of electricity from either metallic surface balanced by air immediately before disruption, in grammes weight per square centimetre.† $\frac{R^2}{8\pi\times981\cdot4}$.
·00254	...	527·7	...	11·290
·00508	...	367·8	...	5·484
·00762	...	314·4	...	4·007
·01016	...	267·6	...	2·903
·01270	...	234·0	...	2·220
·01524	...	216·1	...	1·898
·01778	211·4	208·2	+3·2	1·757
·02032	...	193·1	...	1·512
·02286	...	183·4	...	1·364
·02540	...	177·5	...	1·277
·02667	172·8	173·3	−0·5	1·217
·02794	...	171·0	...	1·185
·02921	166·4	166·9	−0·5	1·129
·03048	...	163·2	...	1·080
·03302	...	159·4	...	1·030
·03556	155·8	155·8	·0	·984
·03810	...	152·6	...	·944
·04064	...	149·9	...	·911
·04318	142·5	144·4	−1·9	·845
·04572	146·7	145·7	+1·0	·860
·06096	142·5	...	...	·828
·07493	129·6	...	...	·681
·08636	126·0	...	...	·644
·09779	121·8	...	...	·601
·10414	123·7	...	...	·620
·11303	121·8	...	...	·601
·12192	119·5	...	...	·579
·13208	116·3	...	...	·548
·13970	115·4	...	...	·540
·14732	114·5	...	...	·531
·15240	114·9	...	...	·535

* Distance between discs of absolute electrometer = ·508 of a centimetre.
Area of each = 174 square centimetres.
Force of gravity at Glasgow on unit mass = 981·4 dynamical units of force; that is to say, generates in one second a velocity of 981·4 centimetres per second.

† This is most directly obtained by finding the force between the discs of the absolute electrometer per square centimetre, and reducing, according to the inverse proportion of squares of distances, to what it would have been if the distance between them had been equal to the length of the spark.

APPENDIX (§§ 327-338).

327. In order that the different expressions, "potential," "electromotive force," "electric force," or "electrostatic force," "pressure of electricity from a metallic surface balanced by air," used in the preceding statement, may be perfectly understood, I add the following explanations and definitions belonging to the ordinary elements of the mathematical theory of electricity :—

328. *Measurement of quantities of electricity.*—The unit quantity of electricity is such a quantity, that, if collected in a point, it will repel an equal quantity collected in a point at a unit distance with a force equal to unity.

[In absolute measurements the unit distance is one centimetre; and the unit force is that force which, acting on a gramme of matter during a second of time, generates a velocity of one centimetre per second. The weight of a gramme at Glasgow is 981·4 of these units of force. The weight of a gramme in any part of the earth's surface may be estimated with about as much accuracy as it can be without a special experiment to determine it for the particular locality, by the following expression :—

In latitude λ, average weight of a gramme

$$= 978{\cdot}024 \times (1 + {\cdot}005133 \times \sin^2\lambda) \text{ absolute kinetic units.}]$$

330. *Electric density.*—This term was introduced by Coulomb to designate the quantity of electricity per unit of area in any part of the surface of a conductor. He showed how to measure it, though not in absolute measure, by his proof plane.

331. *Resultant electric force at any point in an insulating fluid* [compare § 65, above].—The resultant force at any point in air or other insulating fluid in the neighbourhood of an electrified body, is the force which a unit of electricity concentrated at that point would experience if it exercised no influence on the electric distributions in its neighbourhood.

332. *Relation between electric density on the surface of a conductor, and electric force at points in the air close to it.*—According to a proposition of Coulomb's, requiring, however, correction, and first correctly given by Laplace, the resultant force at any point in the air close to the surface of a conductor is perpendi-

cular to the surface and equal to $4\pi\rho$, if ρ denotes the electric density of the surface in the neighbourhood (§ 87, Cor.).

333. *Electric pressure from the surface of a conductor balanced by air.*—A thin metallic shell or liquid film, as for instance a soap-bubble, if electrified, experiences a real mechanical force in a direction perpendicular to the surface outwards, equal in amount per unit of area to $2\pi\rho^2$, ρ denoting, as before, the electric density at the part of the surface considered (§ 88). This force may be called either a repulsion (as according to the views of the eighteenth century school) or an attraction effected by tension of air between the surface of the conductor and the conducting boundary of the air in which it is insulated, as it would probably be considered to be by Faraday; but whatever may be the explanation of the *modus operandi* by which it is produced, it is a real mechanical force, and may be reckoned as in Col. 5 of the preceding Table, in grammes weight per square centimetre. In the case of the soap-bubble, for instance, its effect will be to cause a slight enlargement of the bubble on electrification with either vitreous or resinous electricity, and a corresponding collapse on being perfectly discharged. In every case we may regard it as constituting a deduction from the amount of air-pressure which the body experiences when unelectrified. The amount of this deduction being different in different parts according to the square of the electric density, its resultant action on the whole body disturbs its equilibrium, and constitutes in fact the resultant of the electric force experienced by the body.

334. *Collected formulæ of relation between electric density on the surface of a conductor, electric diminution of air-pressure upon it, and resultant force in the air close to the surface.*—Let, as before, ρ denote the first of these three elements, let p denote the second reckoned in units of force per unit of area, and let R denote the third. Then we have

$$R = 4\pi\rho,$$

$$p = 2\pi\rho^2 = \frac{1}{8\pi}R^2.$$

335. *Electric potential* [difference of potentials being what, after German usage, is still sometimes called "electromotive force." (*Addition, April* 1870.)]—The amount of work required

to move a unit of electricity against electric repulsion from any one position to any other position, is equal to the excess of the electric potential of the second position above the electric potential of the first position.

Cor. 1. The electric potential at all points close to the surface of an electrified metallic body has one value, since an electrified point, possessing so small a quantity of electricity as not sensibly to influence the electrification of the metallic surface, would, if held near the surface in any locality, experience a force perpendicular to the surface in its neighbourhood.

Cor. 2. The electric potential throughout the interior of a hollow metallic body, electrified in any way by external influence, or, if insulated, electrified either by influence or by communication of electricity to it, is constant, since there is no electric force in the interior in such circumstances.

[It is easily shown by mathematical investigation, that the electric force experienced by an electric point containing an infinitely small quantity of electricity, when placed anywhere in the neighbourhood of a hollow electrified metallic shell, gradually diminishes to nothing if the electric point be moved gradually from the exterior through a small aperture in the shell into the interior. Hence the one value of the potential close to the surface outside, mentioned in Cor. 1, is equal to the constant value throughout the interior mentioned in Cor. 2.]

336. *Interpretation of measurement by electrometer.*—Every kind of electrometer consists of a cage or case containing a moveable and a fixed conductor, of which one at least is insulated and put in metallic communication, by what I shall call the principal electrode passing through an aperture in the case or cage, with the conductor whose electricity is to be tested. In every properly constructed electrometer, the electric force experienced by the moveable part in a given position cannot be electrically influenced except by changing the difference of potentials between the principal electrode and the uninsulated conductor or conducting system in the electrometer. Even the best of ordinary electrometers hitherto constructed do not fulfil this condition, as the inner surface of the glass of which the whole or part of the enclosing case is generally made, is liable to become electrified, and inevitably does become so

when any very high electrification is designedly or accidentally introduced, even for a very short time; the consequence of which is that the moving body will generally not return to its zero position when the principal electrode is perfectly disinsulated. Faraday long ago showed how to obviate this radical defect by coating the interior of the glass case with a fine network of tinfoil; and it seems strange that even at the present day electrometers for scientific research, as, for instance, for the investigation of atmospheric electricity, should be constructed with so bad and obvious a defect uncured by so simple and perfect a remedy. When it is desired to leave the interior of the electrometer as much light as possible, and to allow it to be clearly seen from any external position with as little embarrassment as possible, a cage made like a bird's cage, with an extremely fine wire on a metal frame, inside the glass shade used to protect the instrument from currents of air, etc., may be substituted with advantage for the tinfoil network lining of the glass. It appears, therefore, that a properly constructed electrometer is an instrument for measuring, by means of the motions of a moveable conductor, the difference of potentials of two conducting systems insulated from one another, of one of which the case or cage of the apparatus forms part. It may be remarked in passing, that it is sometimes convenient in special researches to insulate the case or cage of the apparatus, and allow it to acquire a potential differing from that of the earth, and that then, as always, the subject of measurement is the difference of potentials between the principal electrode and the case or cage, while in the ordinary use of the instrument the potential of the latter is the same as that of the earth. Hence we may regard the electrometer merely as an instrument for measuring differences of potential between two conducting systems mutually insulated; and the object to be aimed at in perfecting any kind of electrometer (more or less sensitive as it may be, according to the subjects of investigation for which it is to be used), is, *that accurate evaluations in absolute measure, of differences of potential, may be immediately derivable from its indications.*

337. *Relation between electrostatic force and variation of electric potential.*—§ 335, otherwise stated, is equivalent to this:—The average component electrostatic force in the straight line of air between two points in the neighbourhood of an electrified body is equal to their difference of potentials divided by their distance. In other words, the rate of variation of electric potential per unit of length in any direction is equal to the component of the electrostatic force in that direction. Since the average electrostatic force in the line joining two points at which the values of the potential are equal is nothing, the direction of the resultant electrostatic force at any point must be perpendicular to the equipotential surface passing through that point; or the lines of force (which are generally curves) cut the series of equipotential surfaces at right angles. The rate of variation of potential per unit of length along a line of force is therefore equal to the electrostatic force at any point.

338. *Stratum of air between two parallel or nearly parallel plane or curved metallic surfaces maintained at different potentials.*—Let a denote the distance between the metallic surfaces on each side of the stratum of air at any part, and V the difference of potentials. It is easily shown that the resultant electrostatic force is sensibly constant through the whole distance, from the one surface to the other; and being in a direction sensibly perpendicular to each, it must (§ 337) be equal to $\frac{V}{a}$. Hence (§ 332) the electric density on each of the opposed surfaces is equal to $\frac{V}{4\pi a}$. This is Green's theory of the Leyden phial.

339. *Absolute Electrometer.*—As a particular case of § 338, let the discs be plane and parallel: and let the distance between them be small in comparison with their diameters, or with the distance of any part of either from any conductor differing from it in potential. The electric density will be uniform over the whole of each of the opposed surfaces and equal to $\frac{V}{4\pi a}$, being positive on one and negative on the other; and in all other parts of the surface of each the electrification will be comparatively insensible. Hence the force of attraction between them per unit of area (§§ 333 and 334) will be $\frac{V^2}{8\pi a^2}$; if A denote the area of either of the opposed surfaces, the

whole force of attraction between them is therefore $A\dfrac{V^2}{8\pi a^2}$. Hence, if the observed force be equal to the weight of *w* grammes at Glasgow, we have

$$981{\cdot}4 \times w = A\frac{V^2}{8\pi a^2},$$

and therefore

$$V = a\sqrt{\frac{981{\cdot}4 \times 8\pi \times w}{A}}.$$

ADDITION, DATED APRIL 12, 1860.

340. Experiments on precisely the same plan as those of Table I. December 13, have been repeated by the same two experimenters, with different distances from ·75 to 1·5 of a centimetre between the plates of the absolute electrometer, and results have been obtained confirming the general character of those shown in the preceding Tables.

The absolute evaluations derived from these later series must be more accurate than those deduced above from the single series of December 13, when the distance between the plates in the absolute electrometer was only ·5 of a centimetre. I therefore, by permission, add the following Table of absolute determinations:—

Length of spark in centimetres. s.	Electrostatic forces according to estimated average of determinations of February 15, 23, 28, and 29, and March 2. R.
·0086	267·1
·0127	257·0
·0127	262·2
·0190	224·2
·0281	200·6
·0408	151·5
·0563	144·1
·0584	139·6
·0688	140·8
·0904	134·9
·1056	132·1
·1325	131·0

These results, as well as those shown in the preceding Tables, demonstrate a much less rapid variation with distance, of the electrostatic force preceding a spark, at the greater than at the smaller distances. It seems most probable that at still greater distances the electrostatic force will be found to be sensibly constant, as it was certainly expected to be at all distances. The limiting value to which the results shown in the last Table seem to point must be something not much less than 130. This corresponds to a pressure of 68 grammes weight per square decimetre. We may therefore conclude that the ordinary atmospheric pressure of 103,200 grammes per square decimetre, is electrically relieved by the subtraction of not more than 68, on two very slightly convex metallic surfaces, before the air between them is cracked and a spark passes, provided the distance between them is not less than $\frac{1}{6}$ of a centimetre. By taking into account the result of my preceding communication to the Royal Society, we may also conclude that a Daniell's battery of 5510 elements can produce a spark between two slightly convex metallic surfaces at $\frac{1}{6}$ of a centimetre asunder in ordinary atmospheric air.

XX. REPORT ON ELECTROMETERS AND ELECTROSTATIC MEASUREMENTS.

[*British Association Report for* 1867. *Report of Committee on Standards of Electrical Resistance.*]

341. An electrometer is an instrument for measuring differences of electric potential between two conductors through effects of electrostatic force, and is distinguished from the galvanometer, which, of whatever species, measures differences of electric potentials through electromagnetic effects of electric currents produced by them. When an electrometer merely indicates the existence of electric potential, without measuring its amount, it is commonly called an electroscope; but the name electrometer is properly applied when greater or less degrees of difference are indicated on any scale of reckoning, if approximately constant, even during a single series of experiments. The first step towards accurate electrometry in every case is to deduce from the scale-readings, numbers which shall be in simple proportion to the difference of potentials to be determined. The next and last step is to assign the corresponding values in absolute electrostatic measure. Thus, when for any electrometer the first step has been taken, it remains only to determine the single constant coefficient by which the numbers, deduced from its indications as simply proportional to differences of potential, must be multiplied to give differences of potential in absolute electrostatic measure. This coefficient will be called, for brevity, the absolute coefficient of the instrument in question.

342. Thus, for example, the gold-leaf electrometer indicates differences of potential between the gold leaves and the solid walls enclosing the air-space in which they move. If this solid be of other than sufficiently perfect conducting material, of wood and glass, or of metal and glass, for instance, as in the

instrument ordinarily made, it is quite imperfect and indefinite in its indications, and is not worthy of being even called an electroscope, as it may exhibit a divergence when the difference of potentials which the operator desires to discover is absolutely zero. It is interesting to remark (§ 336) that Faraday first remedied this defect by coating the interior of the glass case with tinfoil, cut away to leave apertures proper and sufficient to allow indications to be seen, but not enough to cause these indications to differ sensibly from what they would be if the conducting envelope were completely closed around it; and that not till a long time after did any other naturalist, mathematician, or instrument-maker seem to have noticed the defect, or even to have unconsciously remedied it.

343. Electrometers may be classified in genera and species according to the shape and kinematic relations of their parts; but as in plants and animals a perfect continuity of intermediate species has been imagined between the rudimentary plant and the most perfect animal, so in electrometers we may actually construct species having intermediate qualities continuous between the most widely different genera. But, notwithstanding, some such classification as the following is convenient with reference to the several instruments commonly in use and now to be described:—

I. Repulsion electrometers.
    Pair of diverging straws as used by Beccaria, Volta, and others, last century.
    Pair of diverging gold leaves (Bennet).
    Peltier's electrometer.
    Delmann's electrometer.
    Old station-electrometer, described in lecture to the Royal Institution, May 1860 [§§ 274-275, above]; also in Nichol's *Cyclopædia*, article "Electricity, Atmospheric" (edition, 1860) [§ 263, above], and in Dr. Everett's paper of 1867, "On Atmospheric Electricity" (*Philosophical Transactions*).
II. Symmetrical electrometers.
    Bohnenberger's electrometer.
    Divided-ring electrometers.

Plate I

Fig. 2.

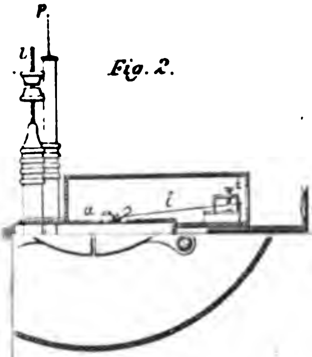
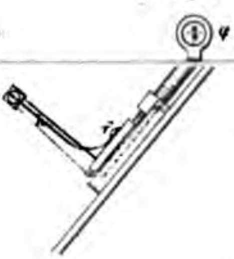


Fig. 7.



J.W. Lowry sculp.t

similar dry piles standing vertically on a horizontal plate of metal, one with its positive and the other with its negative pole up.

† Accademia Pontificia dei Nuovi Lincei.

Bohnenberger's electrometer.
Divided-ring electrometers.

III. Attracted disc electrometers.

- Absolute electrometer.
- Long-range electrometer.
- Portable electrometer.
- Spring-standard electrometer.

344. Class I. is sufficiently illustrated by the examples referred to; and it is not necessary to explain any of these instruments minutely at present, as they are, for the present at all events, superseded by the divided-ring electrometer and electrometers of the third class.

There are at present only two known species of the second class; but it is intended to include all electrometers in which a symmetrical field of electric force is constituted by two symmetrical fixed conductors at different electric potentials, and in which the indication of the force is produced by means of an electrified body moveable symmetrically in either direction from a middle position in this field. This definition is obviously fulfilled by Bohnenberger's well-known instrument.[*]

345. My first published description of a divided-ring electrometer is to be found in the *Memoirs of the Roman Academy of Sciences*[†] for February 1857; but since that time I have made great improvements in the instrument—first, by applying a light mirror to indicate deflections of the moving body; next, by substituting for two half rings four quadrants, and consequently for an electrified body projecting on one side only of the axis, an electrified body projecting symmetrically on the two sides, and moveable round an axis; and lastly, by various mechanical improvements, and by the addition of a simple gauge to test the electrification of the moveable body, and of a replenisher to raise this electrification to any desired degree.

346. In the accompanying drawings, Plate I. fig. 1 represents the front elevation of the instrument, of which the chief bulk consists of a jar of white glass (flint) supported on three legs by a brass mounting, cemented round the outside of its mouth, which is closed by a plate of stout sheet-brass, with

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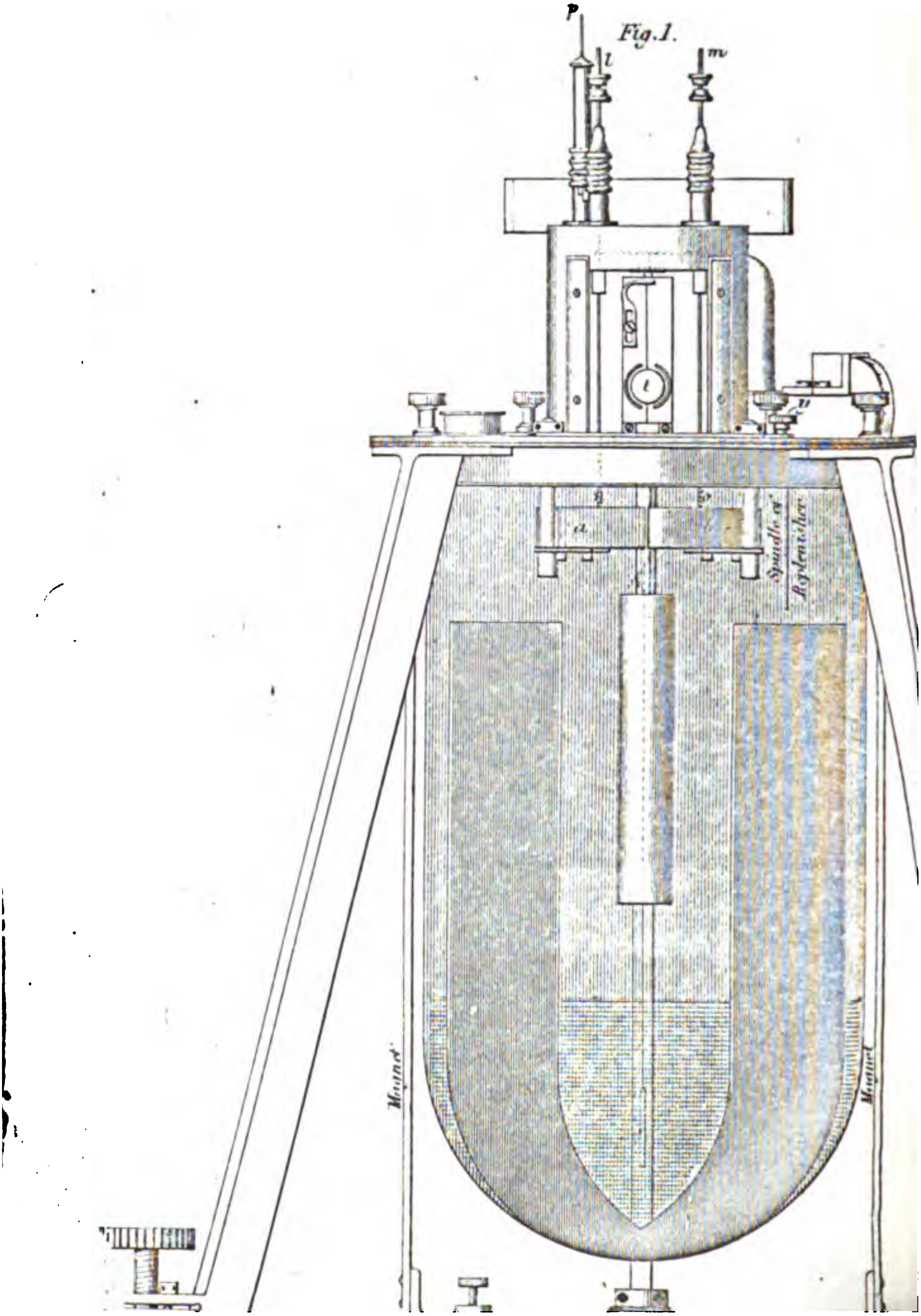
[*] A single gold leaf hanging between the upper ends of two equal and similar dry piles standing vertically on a horizontal plate of metal, one with its positive and the other with its negative pole up.

[†] Accademia Pontificia dei Nuovi Lincei.

Fig. 1.



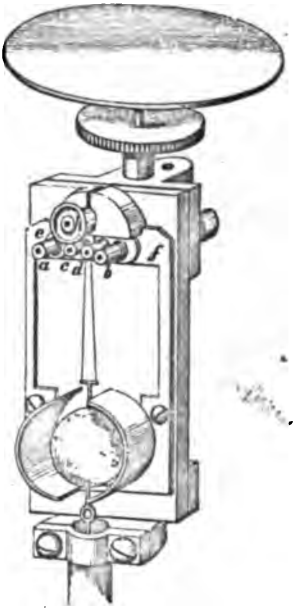
a lantern-shaped cover standing over a wide aperture in its centre. For brevity, in what follows these three parts will be called the jar, the main cover, and the lantern.

Fig. 5 represents the quadrants as seen from above; they are shown in elevation at *a* and *b*, fig. 1, and in section at *c* and *d*, fig. 2. They consist of four quarters of a flat circular box of brass, with circular apertures in the centres of its top and bottom. Their position in the instrument is shown in figs. 1, 2, and 6. Each of the four quadrants is supported on a glass stem passing downwards through a slot in the main cover of the jar, from a brass mounting on the outside of it, and admits of being drawn outwards for a space of about 1 centimetre ($\frac{2}{5}$ of an inch) from the positions they occupy when the instrument is in use, which are approximately those shown in the drawings. Three of them are secured in their proper positions by nuts (*e*, *e*, *e*) on the outside of the chief flat lid of the jar shown in fig. 4. The upper end of the stem, carrying the fourth, is attached to a brass piece (*f*, fig. 6) resting on three short legs on the upper side of the main cover, two of these legs being guided by a straight V-groove at (*g*) to give them freedom to move in a straight line inwards or outwards, and to prevent any other motion. This brass piece is pressed outwards and downwards by a properly arranged spring (*h*), and is kept from sliding out by a micrometer-screw (*i*) turning in a fixed nut. This simple kinematic arrangement gives great steadiness to the fourth quadrant when the screw is turned inwards or outwards, and then left in any position; and at the same time produces but little friction against the sliding in either direction. The opposite quadrants are connected in two pairs by wires, as shown in fig. 5; and two stout vertical wires (*l*, *m*), called the chief electrodes, passing through holes in the roof of the lantern, are firmly supported by long perforated vulcanite columns passing through those holes, and serve to connect the pairs of quadrants with the external conductors whose difference of potentials is to be tested. Springs (*n*, *o*) at the lower ends of these columns, shown in figs. 1 and 2, maintain metallic contact between the chief electrodes and the upper sides of two contiguous quadrants (*a* and *b*) when the lantern is set down in its proper position, but allow the lantern to be removed, carrying the chief electrodes with it, and to be replaced at pleasure without disturbing the quadrants. The lantern also carries an insulated charging-rod (*p*), or temporary electrode, for charging the inner coating of the jar (§ 351) to a small degree, to be increased by the replenisher (§ 352), or, it may be, for making special experiments in which the potential of the interior coating of the jar is to be measured by a separate electrometer, or kept at any stated amount of difference from that of the outer coating. When not in use this temporary electrode is secured in a position in which it is disconnected from the inner coating.

347. The main cover supports a glass column (*q*, fig. 2) projecting vertically upwards through its central aperture, to the upper end of which is attached a brass piece (*r*), which bears above it a fixed attracting disc (*s*), to be described later (§ 353); and projecting down from it a fixed plate bearing the silk-fibre suspension of the mirror (*t*), needle (*u*), etc., seen in figs. 1 and 2, and fixed guard tubes (*v*, *w*), to be described presently. To the main cover also is attached the circular level (fig. 6), which is adjusted to indicate the position of the instrument in which the quadrants are level, and the guard-tubes just mentioned vertical. Its lower surface which rests on the cover is slightly rounded, like a convex lens, so as to admit of a slight further adjustment (see end of § 348, *Addition*) by varying the relative pressure of the three screws by which it is fastened down to the cover.

348. The moveable conductor of the instrument consists of a stiff platinum wire (*x*), about 8 centimetres (3⅛ inches) long, with the needle rigidly attached in a plane perpendicular to it, and connected with sulphuric acid in the bottom of the jar by a fine platinum wire hanging down from its lower end and kept stretched by a platinum weight under the level of the liquid. The upper end of the stiff platinum wire is supported by a single silk-fibre so that it hangs down vertically. The mirror is attached to it just below its upper end. Thus the mirror, the needle, and the stiff platinum stem constitute a rigid body having very perfect freedom to move round a vertical axis (the line of the bearing fibre), and yet practically prevented from any other motion in the regular use of the instrument by the

weight of its own mass and that of the loose piece of platinum hanging from it below the surface of the liquid in the jar. A very small magnet is attached to the needle, which, by strong magnets fixed outside the jar, is directed to one position, about which it oscillates after it is turned through any angle round the vertical axis, and then left to itself. The external magnets are so placed that when there is magnetic equilibrium the needle is in the symmetrical position shown in figs. 5 and 6 with reference to the quadrants.*



[*Addition, April* 1870.—The success of the experiments referred to in the footnote has led to the adoption of the bifilar suspension in all the Quadrant Electrometers now made. It is represented in the margin. The stiff platinum wire which carries the mirror and needle has a cross piece at its upper end, to which are attached the lower ends of the two suspending silk fibres; the other ends being wound upon the two pins *c*, *d*, which may be turned in their sockets by a square-pointed key, to equalize the tensions of the fibres, and make the needle hang midway between the upper and under surfaces of the quadrants. The pins *c*, *d*, are pivoted in blocks carried by springs *e*, *f*, to allow them to be shifted horizontally when adjusting the position of the points of suspension. The screws *a*, *b*, which traverse these blocks, have their points bearing against the fixed plate behind, so that when *a* or *b* is turned in the direction of the hands of a watch, the neighbouring point of suspension is brought forward, and conversely. The needle may thus be made to turn through an angle, till it lies in the symmetrical position represented in fig. 5, Plate I., when all electrical disturbance has been guarded against by connecting the quadrants with the inside and outside of the jar. The conical pin *h* passes between the two springs and screws into the plate behind; by screwing it inwards the points of suspension are made to recede from each other laterally, and the sensibility of the needle to a deflecting couple is diminished, and conversely.

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* Recently I have made experiments on a bifilar suspension with a view to superseding the magnetic adjustment, which promise well.

The method employed to test the symmetry of the suspension is suggested by the consideration that if the tension be equally distributed between the two fibres, the sensibility of the needle to the same deflecting couple will be less than if the whole or the greater part of the weight were supported by one fibre; also, the sensibility being a minimum, a small deviation from the conditions which make it so will produce the least change of sensibility, by the known property of a maximum or minimum. To test whether these conditions are attained, raise first one side of the instrument a little (one turn of the foot-screw on that side is usually sufficient), and then produce an equal deviation in the opposite direction from the position marked by the attached level (§ 347); and in each position of the instrument observe the deflection of the image on the scale produced by some constant difference of potentials, as that between the two poles of a Daniell's cell. This deflection ought to be very nearly equal in the three positions, but exactly equal in the two disturbed positions, and somewhat greater in these than in the middle or level position. When the instrument is far out of adjustment, the deviation will be greater in one of the disturbed positions and less in the other than in the middle position. When it is but slightly out of adjustment, the deflections in the disturbed positions may both somewhat exceed that in the middle position, but to different degrees. An approximation to symmetry thus far at least should be obtained by merely turning the pins (*c*, *d*) in their sockets as already directed, through the minutest angles sensible to the operator, without altering the adjustment of the spirit-level on the cover. When that has been done, the level on the cover ought to be adjusted (§ 347) by successive trials to indicate the position of the instrument such that when equally disturbed from it in opposite directions, the deflections obtained are equally in excess of the deflection obtained in the indicated position.]

349. The needle (*u*) is of thin sheet aluminium cut to the shape seen in figs. 5 and 6; the very thinnest sheet that gives the requisite stiffness being chosen. Its area is 4⅓ square centimetres, and weight ·07 of a gramme. If the four quadrants are in a perfectly symmetrical position round it, and if they are kept at one electric potential by a metallic arc connecting the chief electrodes outside, the needle may be strongly electrified without being disturbed from its position of magnetic equilibrium; but if it is electrified, and if the external electrodes be disconnected, and any difference of potentials established between them, the needle will clearly experience a couple turning it round its vertical axis, its two ends being driven from the positive quadrants towards the negative, if it is itself positively electrified. It is kept positive rather than negative in the ordinary use of the instrument, because I find that when a conductor with sharp edges or points is surrounded by another presenting everywhere a smooth surface, a much greater difference of potentials can be established between them, without producing disruptive discharge, if the points and edges are positive than if they are negative.

350. The mirror (*t*) serves to indicate, by reflecting a ray of light from a lamp, small angular motions of the needle round the vertical axis. It is a very light, concave, silvered glass mirror, being of only 8 millimetres (⅓ of an inch) diameter, and 22 milligrammes (⅓ of a grain) weight. I had for many years experienced great difficulty in getting suitable mirrors for my form of mirror galvanometer; but they are now supplied in very great perfection by Mr. Becker, of Messrs. Elliott Brothers, London. [*Addition, May* 1870.—I have not succeeded in getting more of these light ground concave mirrors giving good images, after a few supplied by Mr. Becker at the time when the report was written. The lightest ground mirrors that Mr. Becker can guarantee to give good images, weigh $\frac{1}{22}$ of a gramme ($\frac{7}{10}$ of a grain). These answer well enough for the electrometers, because the aluminium needle weighing $\frac{1}{14}$ of a gramme ($1\frac{1}{10}$ grain), and being of much greater linear dimensions, its moment of inertia is not largely increased by the addition of a mirror of that weight; and they are preferred for this purpose to the exquisite light mirrors supplied by Mr. White, as being stronger and less liable to warp in being

mounted. But for galvanometers, and especially telegraph-signal galvanometers, it is important that the mirrors be the very lightest possible. The only mirrors suitable for this purpose which I can now obtain are supplied by Mr. White. They give very perfect images, and weigh $\frac{1}{50}$ of a gramme ($\frac{3}{10}$ of a grain) without the magnets, and $\frac{1}{50}$ of a gramme with the magnets attached. Mr. White produces them by cutting out and silvering a large number of circles of the thinnest microscope glass, attaching the magnets (four on the back of each mirror), and finally testing for the image. Out of fifty tried, about ten or fifteen are generally found satisfactory. A mirror may give a good image before the magnets are attached, and become warped out of shape and give a bad image after the magnets have been cemented to it.] The focus for parallel rays is about 50 centimetres (20 inches) from the mirror, and thus the rays of the lamp placed at a distance of 1 metre (or 40 inches) are brought to a focus at the same distance. The lamp is usually placed close behind the vertical screen a little below or above the normal line of the mirror, and the image is thrown on a graduated scale extending horizontally above or below the aperture in the screen through which the lamp sends its light. When the mirror is at its magnetic zero position, the lamp is so placed that its image is, as nearly as may be, in a vertical plane with itself, and not more than an inch above or below its level, so that there is as little obliquity as possible in the reflection, and the line traversed by the image on the screen during the deflection is, as nearly as may be, straight. The distance of the lamp and screen from the mirror is adjusted so as to give as perfect an image as possible of a fine wire which is stretched vertically in the plane of the screen across the aperture through which the lamp shines on the mirror; and with Mr. Becker's mirrors, as with Mr. White's selected galvanometer mirrors, I find it easy to read the horizontal motions of the dark image to an accuracy of the tenth of a millimetre. In the ordinary use of the instrument a white paper screen, printed from a copper-plate, divided to fortieths of an inch, is employed, and the readings are commonly taken to about a quarter of a scale-division; but with a little practice they may, when so much accuracy is desired, be read with considerable accuracy to the tenth of a scale-division. Formerly a slit in front of the lamp was used, but the wire giving a dark line in the middle of the image of the flame is a very great improvement, first introduced by Dr. Everett (in consequence of a suggestion made by Professor P. G. Tait) in his experiments on the elasticity of solids made in the Natural Philosophy Laboratory of Glasgow University.*

351. The charge of the needle remains sensibly constant from hour to hour, and even from day to day, in virtue of the arrangement by which it is kept in communication with sulphuric acid in the bottom of the jar, the outside of the jar being coated with tinfoil and connected with the earth, so that it is in reality a Leyden jar. The whole outside of the jar, even where not coated with tinfoil, is in the ordinary use of the instrument, especially in our moist climate, kept virtually at one potential through conduction along its surface. This potential is generally, by connecting wires or metal pieces, kept the same as that of the brass legs and framework of the instrument. To prevent disturbance in case of strongly electrified bodies being brought into the neighbourhood of the instrument, a wire is either wrapped round the jar from top to bottom, or a cage or network of wire, or any convenient metal case, is placed round it; but this ought to be easily removed or opened at any time to permit the interior to be seen. When the instrument is left to itself from day to day in ordinary use, the needle, connected with the inner coating of the jar as just described, loses, of course, unless replenished, something of its charge; but not in general more than ½ per cent. per day, when the jar is of flint-glass made in Glasgow. On trying similar jars of green glass I found that they lost their charge more rapidly per hour than the white glass jars per month. I have occasionally, but very rarely, found white glass jars to be as defective as those green ones, and it is possible that the defect I

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* A Drummond light placed about 70 centimetres from the mirror gives an image, on a screen about 3 metres distant, brilliant enough for lecture-illustrations, and with sufficient definition to allow accurate readings of the positions on a scale marked by the image of a fine vertical wire in front of the light.

found in the green jars may have been an accident to the jars tested, and not an essential property of that kind of glass.

352. I have recently made the very useful addition of a replenisher to restore electricity to the jar from time to time when required. It consists of (1.) a turning vertical shaft of vulcanite bearing two metal pieces called carriers (*b*, *b*, figs. 17 and 18); (2.) two springs (*d*, *d*, figs. 16 and 18), connected by a metallic arc, making contact with the carriers once every half turn of the shaft, and therefore called connectors; and (3.) two inductors (*a*, *a*) with receiving springs (*c*, *c*) attached to them, which make contact with the carriers once every half turn, shortly before the connecting contacts are made. The inductors (*a*, *a*, figs. 16 and 18) are pieces of sheet metal bent into circular cylindrical shapes of about 120° each; they are placed so as to deviate in the manner shown in the drawing from parts of a cylindrical surface coaxial with the turning-shaft, leaving gaps of about 60° on each side. The diameter of this cylindrical surface is about 15 millimetres (about ⅝ of an inch). The carriers (*b*, *b*, figs. 17 and 18) are also of sheet metal bent to cylindrical surfaces, but not exactly circular cylinders; and are so placed on the bearing vulcanite shaft that each is rubbed by the contact springs over a very short space, about 1 millimetre beyond its foremost edge, when turned in the proper direction for replenishing. The receiving springs (*c*, *c*, figs. 17 and 18) make their contacts with each carrier immediately after it has got fairly under cover, as it were, of the inductor. Each carrier subtends an angle of about 60° at the axis of the turning-shaft. The connecting contacts are completed just before the carriers commence emerging from being under cover of the inductors. The carriers may be said to be under cover of the inductors when they are within the angle of 120° subtended by the inductors on each side of the axis. One of the inductors is in metallic communication with the outside coating of the jar, the other with the inside. Figs. 16, 17, and 18 illustrate sufficiently the shape of carriers and the succession of the contacts. The arrow-head indicates the direction to turn for replenishing. When it is desired to diminish the charge, the replenisher is turned backwards. A small charge having been given to the jar from an independent

source, the replenisher when turned forwards increases the difference of potentials between the two inductors and therefore between the two coatings of the jar connected with them by a constant percentage per half turn, unless it is raised to so high a degree as to break down the air-insulation by disruptive discharge. The electric action is explained simply thus:—The carriers, when connected by the connecting springs, receive opposite charges by induction, of which they deposit large proportions the next time they touch receiving springs. Thus, for example, if the jar be charged positively, the carrier emerging from the inductor connected with the inner coating carries a negative charge round to the receiving spring connected with the outside coating, while the other carrier, emerging from the inductor connected with the outside coating, carries a positive charge round to the receiving spring connected with the inside coating. If the carriers are not sufficiently well under cover of the inductors during both the receiving contacts and the connecting contacts to render the charges which they acquire by induction during the connecting contacts greater than that which they carry away with them from the receiving contacts, the rotation, even in the proper direction for replenishing, does not increase, but, on the contrary, diminishes the charge of the jar. The deviations of the inductors from the circular cylinder, referred to above, have been adopted to give greater security against this failure. A steel pivot fixed to the top of the vulcanite shaft, and passing through the main cover, carries a small milled head (*y*, fig. 1) above, on the outside, which is spun rapidly round in either direction by the finger, and thus in less than a minute a small charge in the jar may be doubled. The diminution of the charge, when the instrument is left to itself for twenty-four hours, is sometimes imperceptible; but when any loss is discovered to have taken place, even if to the extent of 10 per cent., a few moments' use of the replenisher suffices to restore it, and to adjust it with minute accuracy to the required degree by aid of the gauge to be described presently. The principle of the "replenisher" is identical with that of the "doubler" of Bennet. In the essentials of its construction it is the same as Varley's improved form of Nicholson's "revolving doubler."

353. The gauge consists of an electrometer of Class III. The moveable attracted disc is a square portion of a piece of very thin sheet aluminium of the shape shown at *a* in fig. 4. It is supported on a stretched platinum wire passing through two holes in the sheet, and over a very small projecting ridge of bent sheet aluminium placed in the manner shown in the magnified drawing, fig. 3. The ends of this wire are passed through holes in curved springs, shown in fig. 4, and are bent round them so as to give a secure attachment without solder, and without touching the straight stretched part of the wire. The ends of the platinum wire (β, β) are attached by cement to the springs, merely to prevent them from becoming loose, care being taken that the cement does not prevent metallic contact between some part of the aluminium wire and one or both of the brass springs. I have constantly found fine platinum wire rendered brittle by ordinary solder applied to it. The use of these springs is to keep the platinum wire stretched with an approximately constant tension from year to year, and at various temperatures. Their fixed ends are attached to round pins, which are held with their axes in a line with the fibre by friction, in bearings forming parts of two adjustable brass pieces (γ, γ) indicated in fig. 4; these pieces are adjusted once for all to stretch the wire with sufficient force, and to keep the square attracted disc in its proper position. The round pins bearing the stretching springs are turned through very small angles by pressing on the projecting springs with the finger. They are set so as to give a proper amount of torsion tending to tilt the attracted disc (a) upwards, and the long end of the aluminium lever (δ), of which it forms a part, downwards. The downward motion of the long end is limited by a properly placed stop. Another stop (ϵ) above limits the upward motion, which takes place under the influence of electrification in the use of the instrument. A very fine opaque black hair (that of a small black-and-tan terrier I have found much superior to any hitherto tried) is stretched across the forked portion of the sheet aluminium in which the long arm of the lever terminates. Looked at horizontally from the outside of the instrument it is seen, as shown in fig. 7, Plate I., against a white background, marked with two very fine black circles. These sight-plates

in the instruments, as now made by Mr. White, are of the same material as the ordinary enamel watch-dials, with black figures on a white ground. The white space between the two circles should be a very little less than the breadth of the hair. The sight-plate is set to be as near the hair as it can be without impeding its motion in any part of its range; it is slightly convex forwards, and is so placed that the hair is nearer to it when in the middle between the black circles than when in any other part of its range. It is thus made very easy, even without optical aid, to avoid any considerable error of parallax in estimating the position of the hair relatively to the two black circles. By a simple plano-convex lens (ϕ, fig. 2), with the convex side turned inwards, it is easy, in the ordinary use of the instrument, to distinguish a motion up or down of the hair amounting to $\frac{1}{1000}$ of an inch. With a little care I have ascertained, Dr. Joule assisting, that a motion of no more than $\frac{1}{50,000}$ of an inch from one definite central position can be securely tested without the aid of other magnifying power than that given by the simple lens. The lens during use is in a fixed position relatively to the framework bearing the needle, but it may be drawn out or pushed in to suit the focus of each observer. To give great magnification, it ought to be drawn out so far that the hair and sight-plate behind may be but little nearer to the lens than its principal focus, and the observer's eye ought to be at a very considerable distance from the instrument, no less than 20 centimetres (8 inches) to get good magnification; and a short-sighted person should use his ordinary concave eye-lens close to his eye. The reason for turning the convexity of the small plano-convex lens inwards is, that with such a lens so placed, if the eye of the observer is too high or too low, the hair seems to him curved upwards or downwards, and he is thus guided to keep his eye on a level sufficiently constant to do away with all sensible effects of parallax on the position of the hair relatively to the black circles. The framework carrying the stretched platinum wire and moveable attracted disc is above the brass roof of the lantern, in which a square aperture is cut to allow the square portion constituting the short arm of the aluminium balance to be attracted downwards by the fixed attracting disc (§ 347), to be presently described. A side view of the attracting plate, the brass roof of the lantern, the aluminium balance, the sight-plate, the hair, and the plano-convex lens is given in section (fig. 2); also a glass upper roof to protect the gauge and the interior of the instrument below from dust and disturbance by currents of air, to which, without this upper roof, it would be exposed, through the small vacant space around the moveable aluminium square. The fixed attracting disc is borne by a vertical screw screwing into the upper brass mounting (*z*, fig. 2) (§ 347), connected with the inner coating of the Leyden jar through the guard tubes, etc., and is secured in any position by the "jam nut," shown in the drawing at *z*, fig. 2. This disc (*s*) is circular, and about 38 millimetres (1½ inch) in diameter, and is placed horizontally with its centre under the centre of the square aperture in the roof of the lantern. Its distance from the lower surface of the roof and of the moveable attracted disc may be from 2½ to 5 millimetres (from $\frac{1}{10}$ to $\frac{1}{5}$ of an inch), and is to be adjusted, along with the amount of torsion in the platinum wire bearing the aluminium balance-arm, so as to give the proper sensibility to the gauge. The sensibility is increased by diminishing the distance from the attracting to the attracted plate, and increasing the amount of torsion. Or, again, the degree of the potential indicated by it when the hair is in the sighted position is increased by increasing the distance between the plates, or by increasing the amount of torsion. If the electrification of the needle is too great, its proper position of equilibrium becomes unstable; or before this there is sometimes a liability to discharge by a spark across some of the air-spaces. The instrument works extremely well with the needle charged but little less than to give rise to one or both of these faults, and I adjust the gauge accordingly.

354. The strength of the fixed steel directing magnets is to be adjusted to give the desired amount of deflection with any stated difference of potentials maintained between the two chief electrodes, when the jar is charged to the degree which brings the hair of the gauge to its sighted position. In the instruments already made, the deflection * by a single cell of

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* That is to say, the number of scale-divisions over which the luminous image moves when the chief electrodes are disconnected from one another and put in metallic connexion with the two plates of a Daniell's battery.

Daniell's amounts to about 100 scale-divisions (of $\frac{1}{20}$ of an inch each and at a distance of 40 inches), if the magnetic directive force is such as to give a period of vibration equal to about 1·5 seconds, when the jar is discharged and the four quadrants are connected with one another and with the inner coating of the jar. Lower degrees of sensibility may be attained better by increasing the magnetic directing force than by diminishing the charge of the jar. Thus, for instance, when it is to be used for measuring and photographically recording the potential of atmospheric electricity at the point where the stream of the water-dropping collector* breaks into drops, the magnetic directing force may be made from 10 to 100 times greater than that just described. When this is to be done it may be convenient to attach a somewhat more powerful magnetic needle than that which has been made in the most recent instruments where a high degree of sensibility has been provided for. But it is to be remarked that in general the directing-force of the external steel magnets cannot be too strong, as the stronger it is the less is the disturbance produced by magnetic bodies moving in the neighbourhood of the instrument.† In laboratory work, where numerous magnetic experiments are being performed in the immediate neighbourhood, and in telegraph factories where there is constant disturbance by large moving masses of iron, the artificial magnetic field of the electrometer ought to be made very strong. To allow this, and yet leave sufficient sensibility to the instrument, the suspended magnetic needle has been made smaller and smaller, until it is now reduced to two small pieces of steel side by side, 6 millimetres ($\frac{1}{4}$ of an inch) long. For a meteorological observatory all that is necessary is, that the directing magnetic force may be so great that the greatest disturbance experienced in magnetic storms shall not sensibly deflect the luminous image.

355. The sensibility of the gauge should be so adjusted that a variation in the charge of the jar, producing an easily perceived change in the position of the hair, shall produce no

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* See Royal Institution Lecture, May 18, 1860 (§§ 278, 279, above), or Nichol's *Cyclopædia*, article "Electricity, Atmospheric" (Edition 1860) (§ 262, above).

† All embarrassment from this source will be done away with if the bifilar plan be adopted (see § 348, *Addition*).

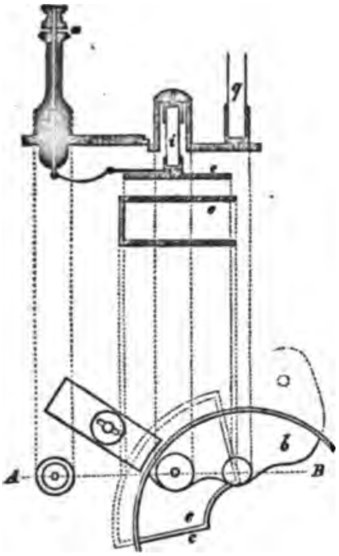
sensible change in the deflection of the luminous image produced by the greatest difference of potentials between the quadrants, which is to be measured in the use of the instrument. I believe the instruments already made, when adjusted to fulfil these conditions, may be trusted to measure the difference of potentials produced by a single cell of Daniell's to an accuracy of a quarter per cent. It must be remembered that the constancy of value of the unit of each instrument depends not only on the constancy of the potential indicated by the gauge, but also on the constancy of the magnetic force in the field traversed by the suspended magnet, and on the constancy of the magnetic moment of the latter. As each of these may be expected to decrease gradually from year to year (although very slowly after the first few hours or weeks), rigorous methods must be adopted to take such variations into account, if the instrument is to be trusted as giving accurately comparable indications at all times. The only method hitherto provided for this most important object consists in the observation of the deflection produced by a measured motion of one of the quadrants by the micrometer screw (*i*) when the four quadrants are put in metallic communication with one another through the principal electrodes; the jar being brought to one constant potential by aid of the gauge, and therefore the force producing the deflection being constant. The amount of the deflection will show whether or not the force of the magnetic field has changed, and will render it easy at any time to adjust the strength of the magnets, if necessary, to secure this constancy. But to attain this object by these means, the three quadrants not moved by the micrometer screw must be clamped by their fixing-screws so that they may be always in the same position.

356. The absolute constancy of the gauge cannot be altogether relied upon. It certainly changes to a sensible degree with temperature; and in different instruments, to very different degrees, and even in different directions, as will be seen (§ 377) in connexion with the description of the portable electrometer to be given later. But this temperature variation does not amount in ordinary cases probably to as much as one per cent.; and it is probable that after a year or two any continued secular variation of the platinum torsion spring will be quite insensible. It is to

be remarked, however, that secular experiments on the elasticity of metals are wanting, and ought at least to be commenced in our generation. In the meantime it will be desirable, both on account of the temperature variation and of the possible secular variation in the couple of torsion, to check the gauge by accurate measurements of the time of oscillation of the needle with its appurtenances. The moment of inertia of this rigid body, except in so far as it may be influenced by oxidation of the metal, of which I have as yet discovered no signs, may be regarded as constant, and therefore the amount of the directing couple due to the magnets may be determined with great accuracy by finding the period of an oscillation when the four quadrants are put in connexion through the charging rod with the metal mounting bearing the guard plates, etc. I have not as yet put into practice any of the obvious methods, founded on the general principle of coincidences used in pendulum observations, for determining the period of the oscillation; but although not more than twenty or thirty complete oscillations can be counted, it seems certain that with a little trouble the period of one of them may be easily determined to an accuracy of about $\frac{1}{10}$ per cent.

357. [*Addition, May* 1870.—The most direct and obvious method of using the Quadrant Electrometer is to connect the two chief electrodes, with the two bodies whose difference of potentials is to be measured, and one of them with the case of the instrument. With the instruments made at the present date, a difference of potentials equal to that of the opposite poles of a single Daniell's cell gives, when measured in this manner, a deflection of the image over about 60 scale-divisions, more or less according to the distance at which the points of suspension of the silk fibres have been adjusted (§ 348, *Addition*). The difference of potentials due to six cells in series would thus deflect the image to the extremity of the scale, and be the greatest difference of potentials that could be measured by the electrometer, if these were the only connexions available for measurements. A second and much lower grade of sensibility is obtained by simply raising, so as to disconnect from the quadrant beneath it, the electrode connected with the case. This being done, it requires a battery of about 10 or 15 cells

to produce the deflection previously produced by a single cell. Several still lower grades of sensibility have been provided for in the instruments recently made, by the addition of an induction-plate, insulated directly over one of the quadrants behind the mirror. The sketch in the margin represents a vertical section through the induction-plate (e), insulating glass stem (i) by which it is supported, its electrode (a), the quadrant (c), and main glass stem (q). The line AB in the horizontal plan below is the line of section, passing through the centres of the electrode and insulating stem of the induction-plate, and that of the main glass stem, which are in one straight line. The plan represents that part of the main cover as seen from above, when the lantern and upper works are removed. The plate (b) which supports the main stem (q) has been enlarged to bear also the insulating support (i) of the induction-plate. The outline of the induction-plate falls within that of the quadrant beneath it by ·16 of a centimetre ($\frac{1}{16}$ of an inch) all round. It is distant ·48 of a centimetre ($\frac{3}{16}$ of an inch) from the upper surface of the quadrant. The dimensions in the figure are half full size.



With an electrometer fitted with the induction-plate, the usual connexions for the first or direct method of measurement are the same as above mentioned. The electrode of the induction-plate may be connected with that of the quadrant beneath it, or with the case, or it may be insulated, without sensibly affecting the indications of the instrument. For the second grade of sensibility the induction-plate is connected with the case, and the difference of potentials to be measured is established between it and the distant pair of quadrants, the nearer pair being insulated by raising their electrode. To free the latter from the induced charge which they commonly receive by the act of raising their electrode, a disinsulator is provided, consisting of a light arm or spring which may be turned so as to make contact with the quadrant by means of a small milled head projecting above the cover. For a certain lower grade the arrangement is the same, except that the distant pair of quadrants, instead of the induction-plate, is connected with the cover, and the difference of potentials to be measured is established between the cover and the induction-plate. With this arrangement the deflections measure about five times the difference of potentials producing the same deflections by the second grade.

The connexions may be further varied so as to produce other degrees of sensibility giving indications perfectly trustworthy and available for comparative measurements. The different methods of forming the connexions, with or without an inductor, are indicated in the following table, where R means the electrode of the pair of quadrants marked RR' in the figure, L that of the pair LL', and I that of the induction-plate; C is the conductor led from one of the bodies experimented upon, O the conductor led from the other and connected to the outer metallic case of the instrument, which may be insulated from the table if necessary by placing a small block or cake of clean paraffin under each of the three feet on which the instrument stands; (R) or (L) means that the electrode of RR' or LL' is to be raised so as to be disconnected from its pair of quadrants. Thus in the grade of diminished power or sensibility standing first in the table on the right, the electrode L is raised, one conductor is connected with R; I and the other with the case of the instrument. The grade standing last in the table, in which L and R are both raised, is the least sensitive of all. In each of these methods the correctness of the indications has been verified by measurements taken simultaneously with the Standard Electrometer (§ 379), the measured difference of potentials being that of the earth and of a Leyden jar fitted with a replenisher, by means of which its potential was varied so as to make the deflected image stand at all points between the extremity of the scale and the zero position. The working of the replenisher being suspended at intervals to allow an accurate reading to be taken of the position of the image and the indication of the Standard Electrometer, the subsistence of a correct proportion between the deflection and the measurement obtained from the Standard Electrometer was verified at all points of the range.

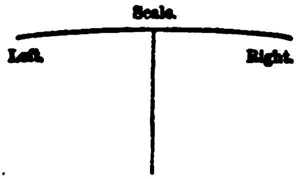
WITHOUT INDUCTOR.

Full Power.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \text{ or } \begin{bmatrix} RC \\ LO \end{bmatrix}$$

Diminished Power.

$$(L)\begin{bmatrix} RC \\ O \end{bmatrix} \text{ or } (R)\begin{bmatrix} LC \\ O \end{bmatrix}$$



WITH INDUCTOR.

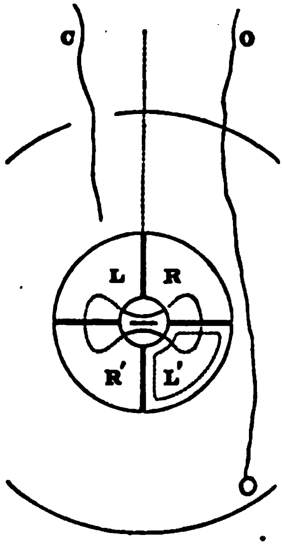
Full Power.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \text{ or } \begin{bmatrix} RC \\ LO \end{bmatrix}$$

Grades of Diminished Power.

$$(L)\left\{\begin{matrix} \begin{bmatrix} RC \\ IO \end{bmatrix} \\ \begin{bmatrix} RIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ RO \end{bmatrix} \end{matrix}\right. \quad (R)\left\{\begin{matrix} \begin{bmatrix} LC \\ IO \end{bmatrix} \\ \begin{bmatrix} LIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ LO \end{bmatrix} \end{matrix}\right.$$

$$(RL)\begin{bmatrix} IC \\ O \end{bmatrix}$$



The facility afforded by the number of these arrangements for varying the sensibility of the instrument even to a moderate or slight degree without altering the adjustment of the fibres, will be found useful in some kinds of observations. For instance, if it be desired to observe the fluctuations of a varying potential, a degree of sensibility which throws the deflected image nearly to the extremity of the scale will cause the fluctuations to be twice as sensible and accurately read as if the deflection were only half as much, as they will bear the same proportion to the whole deflection in the two cases.

It is intended in future to make the induction-plate smaller and more distant from the quadrant, in order to diminish the

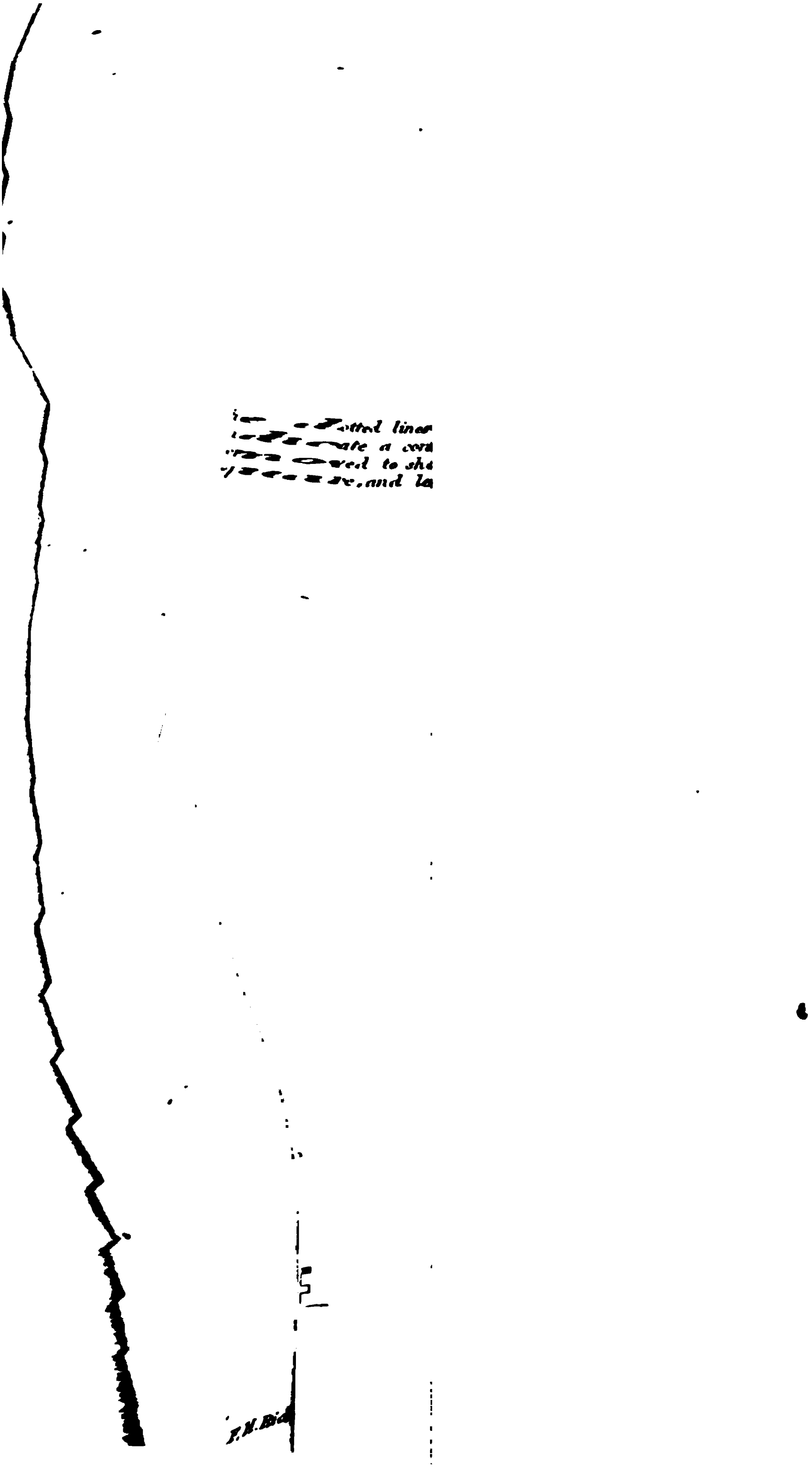

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Fig. 8.

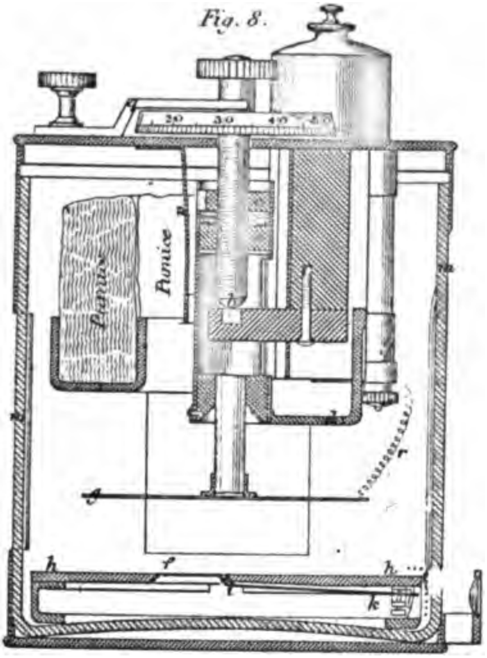
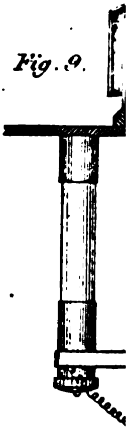


Fig. 9.

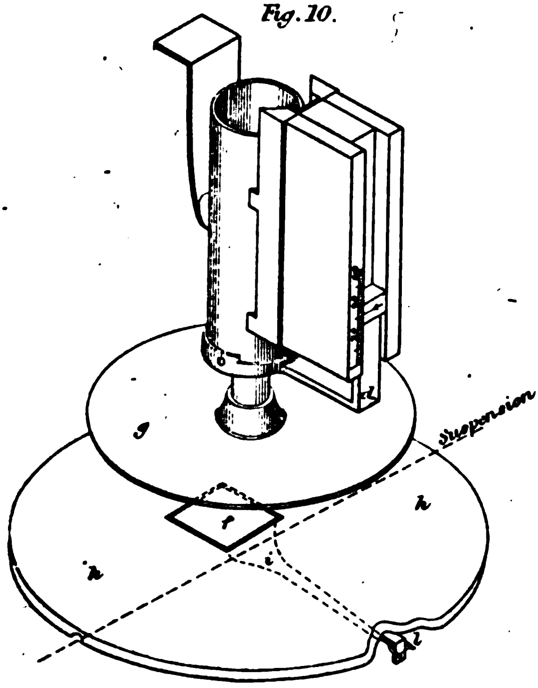


Portable Electr

Figs. 8. 9. & 1

⅔ Full size

Fig. 10.



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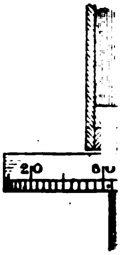

inductive effect and permit of the measurement of from 100 to 5000 cells by the least sensitive method. In some electrometers also the first two grades of sensibility may be considered sufficient, and the induction-plate dispensed with.]

ABSOLUTE ELECTROMETER.

358. The absolute electrometer (fig. 11, Plate II.) and the other instruments of Class III. are founded on a method of experimenting introduced by Sir William Snow Harris, and described in his first paper "On the Elementary Laws of Electricity,"* thirty-four years ago. In these experiments a conductor, hung from one arm of a balance and kept in metallic communication with the earth, is attracted by a fixed insulated conductor, which is electrified, and, for the sake of keeping its electric potential constant, is connected with the inner coating of a Leyden battery. The first result which he announced is, that, when other circumstances remain the same, the attraction varies with the square of the quantity of electricity with which the insulated body is charged and is independent of the unopposed parts. "It is readily seen "that, in the case of Mr. Harris's experiments, it will be "so slight on the unopposed portions that it could not be "perceived without experiments of a very refined nature, such "as might be made by the proof plane of Coulomb, which is, "in fact, with a slight modification, the instrument employed "by Mr. Faraday in the investigation. Now to the degree of approximation to which the electrification of the unopposed parts "may be neglected, the laws observed by Mr. Harris when the "opposed surfaces are plane may be readily deduced from the "mathematical theory. Thus let v be the potential in the interior of A, the charged body, a quantity which will depend "solely on the state of the interior coating of the battery with "which, in Mr. Harris's experiments, A is connected, and will "therefore be sensibly constant for different positions of A "relative to the uninsulated opposed body B. Let a be the "distance between the plane opposed faces of A and B, and

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* *Philosophical Transactions*, 1834.

" let S be the area of the opposed parts of these faces, which " will in general be the area of the smaller, if they be unequal. " When the distance a is so small that we may entirely neglect " the intensity on all the unopposed parts of the bodies, it is " readily shown, from the mathematical theory, that (since the " difference of the potentials at the surfaces of A and B is v) " the intensity of the electricity produced by induction at any " point of the portion of the surface of B which is opposed to " A is $\frac{v}{4\pi a}$, the intensity at any point which is not so situated " being insensible. Hence the attraction on any small element " ω, of the portion S of the surface of B, will be in a direction " perpendicular to the plane and equal to $2\pi\left(\frac{v}{4\pi a}\right)^2\omega$.* Hence " the whole attraction on B is

$$\frac{v^2S}{8\pi a^2}.$$

" This formula expresses all the laws stated by Mr. Harris " as results of his experiments in the case when the opposed " surfaces are plane."†

359. After many trials to make an absolute electrometer founded on the repulsion between two electrified spherical conductors for which I had given a convenient mathematical formula in § 4 of the paper just quoted (§ 30, above), it occurred to me to take advantage of the fact noticed by Harris, but easily seen as an immediate consequence of Green's mathematical theory, that the mutual attraction between two conductors used as in his experiments is but little influenced by the form of the unopposed parts; and in 1853, in a paper " On Transient Electric Currents,"‡ I described a method for measuring differences of electric potential in absolute electrostatic measure founded on that idea. The "absolute electrometer," which I exhibited to the British Association at its Glasgow Meeting in 1855, was constructed for the purpose of putting these methods into practice. This instrument consists of a plane metal disc insulated in a

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* See *Mathematical Journal*, vol. iii. p. 275 (VII. above, §§ 146, 147).

† " On the Elementary Laws of Statical Electricity," *Cambridge and Dublin Mathematical Journal*, 1846; and *Philosophical Magazine*, July 1854 (II. above, § 27).

‡ *Philosophical Magazine*, June 1853.

fixed horizontal position with a somewhat smaller fixed metal disc hung centrally over it, from one end of the beam of a balance. In two papers * entitled "Measurement of Electrostatic Force produced by a Battery," and "Measurement of the Electromotive Force required to produce a Spark in Air between Parallel Metal Plates at Different Distances," published in the *Proceedings of the Royal Society* for February 1860, I described applications of this electrometer, in which, for the first time I believe, absolute electrostatic measurements were made. The calculations of differences of potential in absolute measure were made according to the formula quoted above (§ 358) from my old paper on "The Elementary Laws of Statical Electricity."

360. This formula is rigorous only if the distance between the discs is infinitely small in comparison with their diameters; and therefore, in my earliest attempt to make absolute electrostatic measurements, I used very small distances. I found great difficulty in securing that the distance should be nearly enough equal between different parts of the plates, and in measuring its absolute amount with sufficient accuracy; and found besides serious inconveniences in respect of sensibility and electric range: later I made a great improvement in the instrument by making only a small central area of one of the discs moveable. Thus the electric part of the instrument becomes two large parallel plates with a circular aperture in one of them, nearly filled up by a light circular disc supported properly to admit of its electrical attraction towards the other being accurately measured in absolute units of force. The disc and the perforated plate surrounding it will be called, for brevity, the disc and the guard-plate. The faces of these two next the other plate must be as nearly as possible in one plane when the disc is precisely in the position for measuring the electric force upon it, which, for brevity, will be called its sighted position. The space between the disc and the inner edge of its guard-ring must be a very small part of the diameter of the aperture, and must be very small in comparison with the distance between the plates; but the diameter of the disc may be greater than, equal to, or less than the distance between the plates.

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* XVIII. and XIX. above, §§ 310-340.

361. Mathematical theory shows that the electric attraction experienced by the disc is the same as that experienced by a certain part of one of two infinite planes at the same distance, with the same difference of electric potentials, this area being very approximately the mean between the area of the aperture and the area of the disc, and that the approximation is very good, even should the distance between the plates be as much as a fourth or fifth, and the diameter of the disc as much as three-fourths of the diameter of the smaller of the two plates. This conclusion will be readily assented to when we consider that[*] the resultant electric force at any point in the air between the two plates is equal numerically to the rate of conduction of heat per unit area across the corresponding space in the following thermal analogue. Let a solid of uniform thermal conductivity replace all the air between and around the plates; and in place of the plates let there be hollow spaces in this solid. Let these hollow spaces be kept at two uniform temperatures, differing by a number of degrees equal numerically to the difference of potentials in the electric system, the space corresponding to the disc and guard-ring being at one temperature, and that corresponding to the opposite plate at the other temperature; and let the thermal conductivity of the solid be unity. If we attempt to draw the isothermal surfaces between the hollow corresponding to the continuous plate on the one side, and that corresponding to the disc and guard-ring on the other, we see immediately that they must be very nearly plane, from very near the disc all the way across to the corresponding central portion of the opposite plate, but that there will be a convexity towards the annular space between the disc and guard-ring.

362. Thus we see that the resultant electric force will, to a very close approximation, be equal to $\frac{V}{D}$ for all points of the air between the plates at distances from the outer bounding edges exceeding two or three times the distance between the plates, and at distances from the interstice between the guard-

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[*] "On the Uniform Conduction of Heat through Solid Bodies, and its connexion with the Mathematical Theory of Electricity," *Cambridge Mathematical Journal*, Feb. 1842; and *Philosophical Magazine*, July 1854 (I. above, §§ 1-6).

ring and disc not less than the breadth of this interstice. Hence, if ρ denote the electric density of any point of the plate or disc far enough from the edges, we have

$$\rho = \frac{V}{4\pi D}.$$

But the outward force experienced by the surface of the electrified conductor per unit of area at any point is $2\pi\rho^2$, and therefore if F denote the force experienced by any area A of the fixed plate, no part of which comes near its edge, we have

$$F = \frac{V^2 A}{8\pi D^2},$$

which will clearly be equal to the attraction experienced by the moveable disc, if A be the mean area defined above. This gives $V = D\sqrt{\frac{8\pi F}{A}}$, the formula by which difference of potentials in absolute electrostatic measure is calculated from the result of a measurement of the force F, which, it must be remembered, is to be expressed in kinetic units. Thus if W be the mass in grammes to which the weight is equal, we have

$$F = gW,$$

where g is the force of gravity in centimetres per second per second.

The difficulty which, in first applying this method about twelve years ago, I found in measuring accurately the distance D between the plates and in avoiding error from their not being rigorously parallel, I now elude by measuring only *differences* of distance, and deducing the desired results from the difference of the corresponding differences of potentials. Thus let V' be the difference of potentials between the plates required to give the same force F; when the difference of potentials is V' instead of V, we have

$$V' - V = (D' - D)\sqrt{\frac{8\pi F}{A}}.$$

363. The plan of proceeding which I now use is as follows: —Each plate (fig. 11, Plate II.) is insulated; one of them, the continuous one, for instance, is kept at a potential differing from the earth by a fixed amount tested by aid of a separate

idiostatic[*] electrometer;[†] the other plate (the guard-ring and moveable disc in metallic communication with one another) is alternately connected with the earth and with the body whose potential is to be measured. The lower plate is moved up or down by a micrometer screw until the moveable disc balances in a definite position, indicated by the hair (with background of white with black dots) seen through a lens, as shown in fig. 11. Before and after commencing each series of electrical experiments, a known weight is placed on the disc, and a small wire rider on the lever from which the disc hangs is adjusted to bring the hair to its sighted position when there is no electric force. This last condition is secured by putting the two plates in metallic communication with one another. For the electric experiments the weight is removed, so that when the hair is in the sighted position the electric attraction on the moveable disc is equal to the force of gravity on the weight. The electric connexions suitable in using this instrument for determining in absolute electrostatic measure the difference of potentials maintained by a galvanic battery between its two electrodes are indicated in fig. 11. No details as to the case for preventing disturbance by currents of air, and for maintaining a dry atmosphere, by aid of pumice impregnated with strong sulphuric acid, are shown, because they are by no means convenient in the instrument at present in use, which has undergone so many transformations that scarcely any part of the original structure remains. I hope soon to construct a compact instrument convenient for general use. The amount of force which is constant in each series of experiments may be varied from one series to another by changing the position of the small wire rider on the lever.

The electric system here described is heterostatic (§ 385 below), there being an independent electrification besides that whose difference of potential is to be measured.

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[*] See § 385, below.

[†] [A Leyden jar with an idiostatic gauge and replenisher fitted to the cover by which it is closed has been found very suitable for this purpose. The gauge can be adjusted to a higher degree of sensibility than is attainable in an electrometer for general purposes, as the Standard or the Portable Electrometer, and the micrometer movements and graduations of these electrometers are not required.—*May* 1870.]

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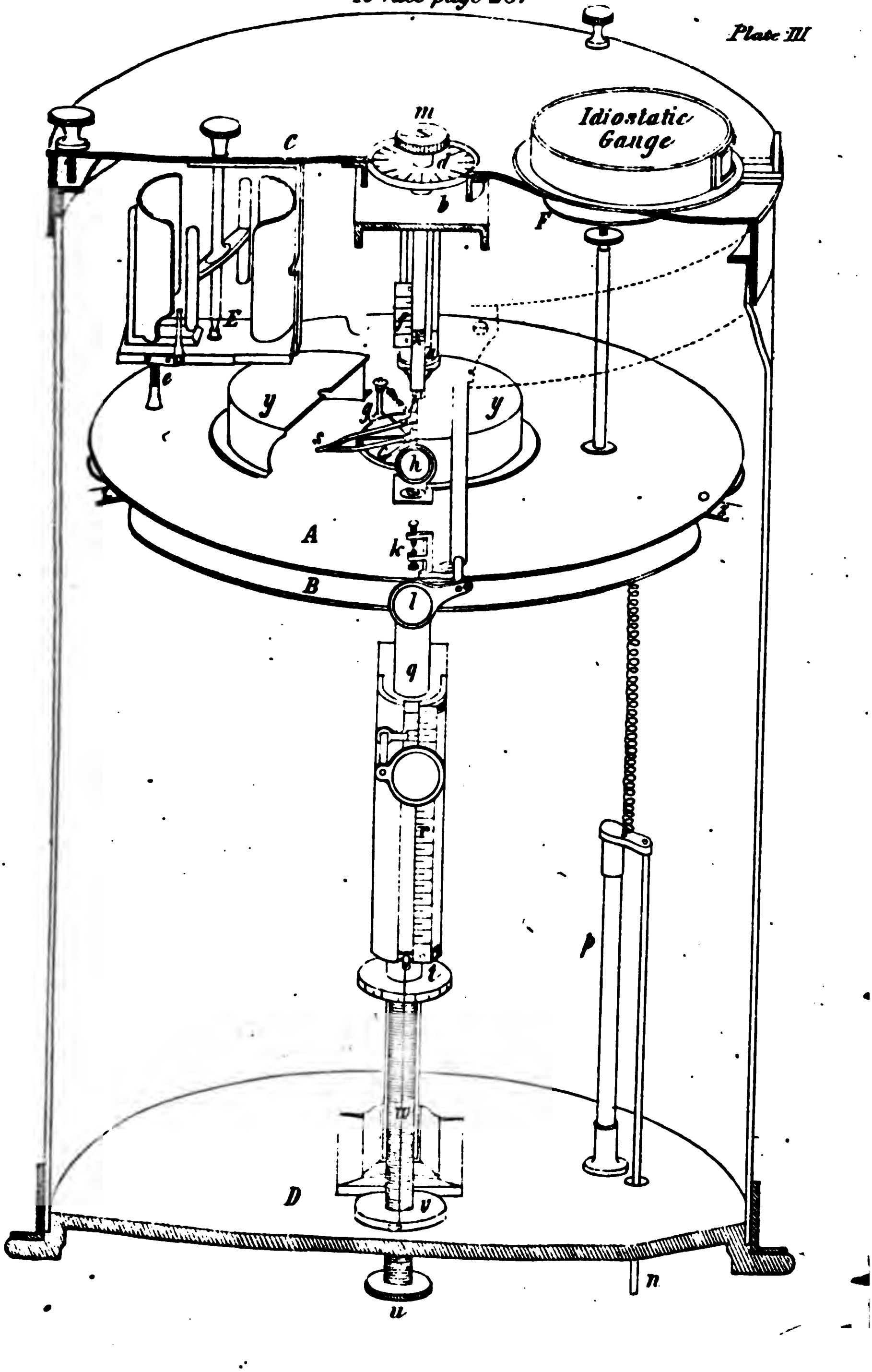
Plate III



NEW ABSOLUTE ELECTROMETER.

[§ 364...367 *added May* 1870.]

364. Plate III. is a sketch in perspective of this instrument, one-third of the full size. As in the Absolute Electrometer just described, the electric system is heterostatic; with this addition, that the potential of the auxiliary charge is tested and maintained, not by a separate electrometer and electric machine, but by an idiostatic arrangement forming part of the instrument itself. This consists of a Leyden jar, forming the case of the instrument; a gauge; and a replenisher. The Leyden jar is a white (flint) glass cylinder, coated inside and outside with tinfoil to nearly the height of the circular plate (*A*); apertures being left to admit the requisite light to the interior, and allow the indications of the vertical scale (*r*) and divided circle (*t*) to be read. A brass mounting is cemented round the upper rim of the jar, to which is screwed the cover of stout sheet-brass (*C*), which closes the jar at the top. By another brass mounting cemented round its lower rim, the jar is fastened down to the cast-iron sole-plate (*D*) which closes its lower end. The sole-plate is supported on three legs similar to those shown in fig. 13, Plate II. The cover (*C*) supports the replenisher (*E*), and the aluminium balance-lever of the idiostatic gauge, which are identical in construction with those described in §§ 352, 353, but on a larger scale. The air inside is kept dry by aid of pumice soaked with strong sulphuric acid, contained in glass vessels placed in the bottom of the jar.

The moveable disc or balance (*c*) hangs in a circular aperture in the plate (*A*), which rests on three fixed supports (*z*, *z*, .) cemented to the interior surface of the jar, and in metallic connexion with the inside coating; the manner of support is that of the hole, slot, and plane, described in § 380, (2), below. This perforated plate or guard-plate supports on a brass pillar the attracting plate (*F*) of the idiostatic gauge, which thus tests the potential of the guard-plate, balance, and inside coating. This potential is kept constant during any series of experiments by using the replenisher according to the indications of the gauge, which is made extremely sensitive by a proper adjustment of the distance from the attracting plate (F) to the balance-lever and of the torsion by which the electrical attraction is balanced (see end of § 353). The replenisher has metallic contact with the guard-plate through the spring (e). The jar is charged by an insulated charging-rod let down for the occasion through a hole in the cover.

365. The balance (c) is a light aluminium disc, about 46 millimetres in diameter, strengthened by an elevated rim and radial ribs on its upper surface, but having its lower surface plane and smooth. It nearly fills the aperture in the guard-plate, sufficient clearance being left (·75 of a millimetre all round) to allow it to move up and down without risk of friction. It is supported by three delicate steel springs, each of which consists of two parts; the upper end of the upper part is attached to the lower extremity of a vertical insulating stem (i) directly above the centre of the disc, where the corresponding end of the lower part is fixed. The opposite ends, which project considerably beyond the circumference of the disc, are riveted together. One of these springs (s) is shown in the figure. Their general form may be compared to that of coach-springs. The point of attachment of their upper parts is moved vertically by a kinematic arrangement precisely the same as that employed in the Portable Electrometer (§ 369). The insulating stem (i) is attached to a brass tube (a), which slides up and down in V guides by the action of a micrometer screw. This micrometer screw is worked by means of the milled head (m) projecting above the cover (C); the guides for the tube (a) and index (x) which moves up and down with the tube, are similar to those represented more fully in fig. 10, Plate II., and are rigidly attached to a strong brass plate (b) lying across the mouth of the jar below the cover, and resting upon the flange of the brass mounting, to which it is fastened by screws. The plate (b) is so adjusted that the balance may hang concentric with the perforation in the guard-plate. The tube (a) is similar in construction to that represented in fig. 8, Plate II., and described in § 369, below. The micrometer screw carries a horizontal circular disc (d) graduated by 100 equal angular divisions. An aperture is left in the cover

through which its indications can be read off by reference to a fixed mark on the sloping edge of the aperture. This, together with the scale (*f*), each division of which corresponds to one full turn of the micrometer screw, measures the vertical distance through which the tube (*a*) and the points of attachment of the springs are moved.

Metallic communication between the balance and the guard-plate is maintained by a light spiral wire attached to the pillar (*g*) and to the upper support of the springs, which is a brass piece cemented to the insulating stem. An arm, not seen in the figure, projects from the guard-plate over the disc so that its extremity is between the centre of the disc and the upper end, bent horizontally, of an upright fixed to the disc; thus serving as a stop to confine the motion of the disc between certain limits. A very fine opaque black hair (§ 353) is stretched between two small uprights (one of which is seen in the figure) standing in the centre of the disc. An achromatic convex lens (*h*), fixed on the guard-plate, stands opposite, and produces an image of the hair in the conjugate focus, which is just over the outer edge of the guard-plate. The two opposed screw-points (*k*) are adjusted to touch each side of the image thus thrown by the lens, which, on the principle of the astronomical telescope, is observed through an eye-lens (*l*), attached outside of the jar to the upper brass mounting. By this arrangement the error of parallax in observing the position of the hair relatively to the two points is avoided; the position of the eye may be varied in any direction without causing any change in the apparent relative position of the hair (image) and points. In adjusting these different parts, it is arranged that when the image of the hair is exactly between the two points, or in what is called the sighted position, the under surfaces of the balance and guard-plate may be as nearly as possible in one horizontal plane.

The balance and springs are protected, in the use of the instrument, from disturbing electrical forces, by a brass cover in two halves (*y*, *y*), one of which is represented displaced in the figure, to show the interior arrangements. The two halves, when placed together, form a circular box, with an aperture in front in which the lens (*h*) stands, and another aperture behind to admit light from the sky or from a lamp placed outside of the jar in the line of the hair, lens, and points.

366. The electrical part of the instrument is completed by the continuous attracting plate (*B*), under and parallel to the guard-plate and spring-balance. This is a stiff circular brass plate with parts cut out to allow it to move freely past the fixed supports (*z*, *z*, .) of the guard-plate. An electrode (*n*) projecting through a hole in the sole-plate from an insulating stem (*p*) is kept in metallic communication by a spiral wire with an arm projecting from the centre of the continuous plate. The plate (*B*) is supported by a brass pillar (*q*), from which it is insulated by a short glass stem. It is moved vertically by the micrometer screw (*w*) (step $\frac{1}{50}$ of an inch); and this motion is measured by a vertical scale (*r*) and horizontal graduated circle (*t*) attached to the screw. The screw projects below the sole-plate, and is worked by the milled head (*u*), the nut (*v*) being fixed in the centre of the sole-plate. The pillar (*q*) moves in V or ring guides, and rests upon the upper end of the screw in the manner represented in fig. 14, Plate II.

367. Before this instrument is available for absolute electrostatic measurements, the force required to move the balance through any fixed vertical distance (the point of suspension being unmoved) must be known. This is ascertained by weighings conducted in the following manner:—The cover (*C*) is removed, and all electrical force upon the balance is guarded against by putting the electrode (*n*) in metallic communication with the guard-plate. The balance is then brought, by turning the micrometer circle (*d*), to the sighted position; and the reading on the scale (*f*) and graduated circle (*d*) is noted. A known weight is then distributed symmetrically over the disc ($\frac{6}{10}$ of a gramme has been used hitherto), which displaces it below the sighted position. It is now raised to the sighted position by turning the disc (*d*), and the altered micrometer reading is noted. The difference between the two readings measures the distance through which the given weight displaces the balance in opposition to the tension of the springs; and conversely, when the balance has been displaced through the same distance by electrical attraction between it and the continuous plate below it, this known weight is the measure of the force exerted

upon it. It has been thus found by repeated weighings, that a weight of $\frac{6}{10}$ of a gramme displaces the balance through a distance corresponding to two full turns of the micrometer screw and a fraction of one division of the circle, in the instrument belonging to the Laboratory of the Glasgow University. This distance having been ascertained with all possible care and at different temperatures, in view of the possible effect of temperature on the elasticity of the springs, the plan of proceeding to absolute electrostatic measurements is as follows, the weights being removed and covers (*y*, *y*, *C*) replaced.

All electrical influence having been removed by a wire led from the electrode (*n*) through the hole in the cover (*C*) to the guard-plate, the balance is brought to the sighted position. Starting from this point, it is raised by the micrometer screw through any distance which has been ascertained to correspond to a known weight, *e.g.* the distance just mentioned. This corresponds exactly to the removal of the weight (§ 363) in the use of the Absolute Electrometer already described. The jar is then charged, and the potential is kept constant during the experiments by using the replenisher according to the indications of the gauge, which, as already said, has been made extremely sensitive for the purpose. The attracting plate (*B*) is connected by its electrode (*n*) alternately with the outside coating of the jar (which may be either connected with the earth or insulated) and with the body the difference of whose potential from that of the outside coating is to be measured. In each case the balance is brought to the sighted position by moving the plate (*B*) up or down by the micrometer screw (*w*), and the reading on the vertical scale (*r*) and graduated circle (*t*) is noted. The difference of the two readings gives the difference of the two distances between balance and attracting plate, from which the difference of potentials is deduced by the formula at the end of § 362. In measuring the difference of potentials between the poles of a voltaic battery, it is found very convenient to connect the poles, through a Steinheil (or double Bavarian) key, either with the outer coating of the jar (or earth), the other with the insulated electrode (*n*). The reading being taken and the key reversed, the difference of readings, it is evident, measures a difference of potentials double that of the poles of the battery. Two observers are convenient, one to watch the gauge and use the replenisher accordingly, the other to take the readings.

PORTABLE ELECTROMETER.

368. In the ordinary use of the portable electrometer (figs. 8, 9, and 10, Plate II.), the electric system is heterostatic and quite similar to that of the absolute electrometer, when used in the manner described above in § 363. But the balance is not adapted for absolute measure of the amount of force of attraction experienced by the moveable disc; on the contrary, it is precisely the same as that described for the gauge of the quadrant electrometer in § 353 above, only turned upside down. Thus, in the portable instrument, the square disc (*f*) forming part of the lever of thin sheet aluminium is attracted *upwards* by a solid circular disc of sheet-brass (*g*), thick enough for stiffness. Every part of the aluminium lever except this square portion is protected from electric attraction by a fixed brass plate (*h h*) with a square hole in it, as nearly as may be stopped by the square part of the sheet aluminium destined to experience the electric attraction, all other parts of the aluminium balance-lever being below this guard-plate. The aluminium lever (*i k*), as shown in figs. 8 and 10, is shaped so that when the hair (*l*) at the end of its long arm is in its sighted position, the upper surfaces of the fixed guard-plate (*h*) and moveable aluminium square (*f*) are as nearly as may be in one plane. The mode of suspension is precisely the same as that described (§ 353) for the gauge of the quadrant electrometer. In the portable instrument, careful attention is given by the maker to balance the aluminium lever by adding to it small masses of shellac or other convenient substance, so that its centre of gravity may be in the line of its platinum-wire axis, or, more properly speaking, in such a position that the instrument shall give, when electrified, the same "earth-readings" when held in any positions, either upright, or inclined, or inverted (§ 375 below). Thus the condition of equilibrium of the balance, when the hair is in its sighted position, is that the moment of electric attraction round the axis of suspension shall

be equal to the moment of the couple of torsion, the latter being as constant as the properties of the matter concerned (platinum wire, brass stretching-springs, etc.) will allow.

369. The guard-plate carrying, by the platinum-wire suspension, the aluminium balance, is attached to the bottom of a small glass Leyden jar (*m m*), and is in permanent metallic communication with its inside coating of tinfoil. The outside tinfoil coating of this jar is in permanent metallic communication with the outside brass protecting case. The upper open mouth of this case is closed by a lid or roof, which bears on its inner side a firm frame projecting downwards. This frame has two V notches, in which a stout brass tube (*o*) slides, kept in the Vs by a properly placed spring (*p*) [(*May* 1870) better two springs, one pressing directly towards each V], giving it freedom to slide up and down in one definite line.* Firmly fixed in the upper end of this tube is a nut (*a*, fig. 8), which is made to move up and down by a micrometer screw. The lower end of the shaft of this screw has attached to it a convex piece of polished steel (*b*, fig. 8), which is pressed upon a horizontal agate plate rigidly attached to the framework above mentioned by a stiff brass piece projecting into the interior of the brass tube through a slot long enough to allow the requisite range of motion. This arrangement will be readily understood from the accompanying drawings. It has been designed upon obvious geometrical principles, which have been hitherto neglected, so far as I know, in all micrometer screw mechanisms, whether for astronomical instruments or other purposes. The screw-shaft is turned by a milled head, fixed to it at the top outside of the roof of the instrument; and the angles through which it is turned are read on a circle divided into one hundred equal parts of the circumference (or 3°·6 each) by reference to a fixed mark

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* In consequence of suggestions by Mr. Jenkin, it is probable that the spring may be done away with, and the Vs replaced by rings approximately fitting round the tube, but leaving it quite free to fall down by its own weight. In consequence of the symmetrical position of the convex end of the screw over the centre of the attracted disc, slight lateral motions of the tube produce no sensible effect on the electric attraction. [(*May* 1870.— Various trials both on the portable and stationary instruments have but very partially fulfilled this anticipation; and have confirmed the practical value of the Vs. The constructional advantages of the rings and geometrical merits of the Vs are easily combined.]

on the roof of the instrument. The hole in the roof through which the screw-shaft passes is wide enough to allow the shaft to turn without touching it, and the lower edge of the graduated circle turning with the screw is everywhere very near the upper side of the roof, but must not touch it at any point. A second nut (*c*, fig. 8) above the effective nut fits easily, but somewhat accurately, in the hollow brass tube, and is prevented from turning round in the tube by a proper projection and slot. Thus the screw is rendered sufficiently steady, with reference to the sliding tube; that is to say, its axis is prevented from any but excessively small deviations from the axis of the sliding tube and fixed guides; and when the nut is kept from being turned round its proper axis, it forms along with the sliding tube virtually a rigid body. A carefully arranged spiral spring presses the two nuts asunder, and so causes the upper side of the thread of the screw-shaft always to press against the under side of the thread of the effective nut, thus doing away with what is technically called in mechanics "lost time." In turning the micrometer screw, the operator presses its head gently downwards with his finger, to secure that its lower end bears firmly upon the agate plate. It would be the reverse of an improvement to introduce a spring attached to the roof of the instrument outside to press the screw head downwards, inasmuch as however smooth the top of the screw-shaft might be made, and however smooth the spring pressing it down, there would still be a very injurious friction impeding the proper settlement of the sliding tube into its Vs. A stiff fork (*g*) stretching over the graduated circle is firmly attached to the roof outside, to prevent the screw from being lifted up by more than a very small space; about $\frac{1}{50}$ of an inch at most. In using the instrument, the observer should occasionally pull up the screw-head and press it down again, and give it small horizontal motions, to make sure that when it is being used it is pressed in properly to its Vs and down upon the agate-plate. A long arm (*d*, figs. 8 and 10) (or two arms one above the other), firmly attached to the sliding-tube, carries an index which moves up and down with it. Two fixed guiding-cheeks on each side of this index prevent the tube from being carried round too far in either direction when the screw is turned: one of these cheeks is graduated so that each division is equal in length to the step of the micrometer screw; this enables the operator to ascertain the number of times he has turned the screw. These two cheeks must never simultaneously press upon the sliding-pointer; on the contrary, they must leave it a slight amount of lateral freedom to move. If this does not amount to ·36 of a degree, the amount of "lost time" produced by it will not exceed $\frac{1}{10}$ of a division of the micrometer circle, and will not produce any sensible error in the use of the instrument. A glass rod cemented to the lower end of the tube prolongs its axis downwards, and bears the continuous attracting-plate of the electrometer at its lower end.

The object aimed at in the mechanism just described is to prevent the nut and other parts rigidly connected with it from any other motion than parallel to one definite line, and to leave it freedom to move in this line, unimpeded by any other friction than that which is indispensable in the arrangement for keeping the sliding tube in its Vs.

370. If the inner tinfoil covering of the Leyden jar were completed up to the guard-plate bearing the aluminium balance-lever, the long arm of this lever being in the interior of a hollow conductor would experience no electric influence, and no force from the electrification of the Leyden jar, or from separate electrification of the upper attracting plate, or, more strictly speaking, the electric density and consequent electric force on the long arm of the lever would be absolutely insensible to the most refined test we could apply, because of the smallness of the gap between the moveable aluminium square and the boundary of the square aperture in the guard plate. But to see the hair on the long end of the lever, and the white background with black dots behind it, a not inconsiderable portion of the glass under the guard-plate must be cleared of tinfoil outside and inside. Thus the electric potential of the inner coating of the Leyden jar will not be continued quite uniformly over the inner surface of the bared portion of the glass, and a disturbance affecting chiefly the most sensitive part of the lever will be introduced. To diminish this as much as possible without inconveniently impeding vision, a double screen of thin wire fencing, in metallic communication with

the inner tinfoil coating and the guard-plate, is introduced between the end of the lever and the glass through which it is observed.

371. A very light spiral spring (*r*) connects the upper attracting plate with a brass piece supported upon a fixed vertical glass column projecting downwards from the roof of the instrument. This brass piece bears a stout wire (*s*), called the main electrode, projecting vertically upwards along the axis of a brass tube open at each end, fixed in an aperture in the roof so as to project above and below, as shown in fig. 9.

372. The top of the main electrode bears a brass sliding piece (*t*), which, when raised a little, serves for umbrella and wind-guard without disturbing the insulation; and when pressed down closes the aperture and puts the electrode in metallic connexion with the roof of the instrument. When the instrument is to be used for atmospheric electricity (unless at a fixed station), a steel wire, about 20 centimetres long, is placed in the hole on the top of the sliding brass piece just mentioned, and is thus held in the vertical position. A burning match is attached to its upper end, which has the effect of bringing the potential of the chief electrode and upper attracting plate, etc., all to the potential of the air at the point where the match burns.[*] The instrument is either held in the observer's hand, or it is placed upon a fixed support, and care taken that its outer brass case is in connexion with the earth. When the difference of potentials between two conductors is to be tested, one of these is connected with the brass case of the instrument, and the other with the chief electrode, the umbrella being kept up. If both of these conductors must be kept insulated from the earth, the brass case of the electrometer must be put on an insulating stand, and the micrometer screw turned by an insulating handle.

373. A lead cup (*e e*, fig. 8), supported by metal pillars from the roof and carrying pieces of pumice-stone, held in their place by India-rubber bands, completes the instrument. The inner surface of the glass must be clean, and particles of dust,

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[*] See Nichol's *Cyclopædia*, article "Electricity, Atmospheric," 2d edition, 1860 (§ 266, above); or "Royal Institution Lecture on Atmospheric Electricity," May 1860 (§§ 277, 278, above).

minute shreds or fibres, etc., removed as carefully as possible, especially from the lower surface of the upper attracting-plate, and the upper surface of the guard-plate and aluminium square facing it from below. The pumice is prepared by moistening it with a few drops of strong pure sulphuric acid. Ordinary sulphuric acid of commerce should be boiled with sulphate of ammonia to free it from volatile acid vapours, and to strengthen it sufficiently by removing water if the acid be not of the strongest. There should not be so much acid applied to the pumice as to make it have the appearance of being moist, but there must be enough to maintain a sufficiently dry atmosphere within the instrument for very perfect insulation of the Leyden jar, which I find does not in general lose more of its charge than five per cent. per week, when the pumice is properly impregnated with acid. Thus there is no tendency of the liquid to drop out of the pumice; and the pumice being properly secured by the India-rubber bands, the instrument may be thrown about with any force, short of that which might break the glass jar or either of the glass stems, without doing any damage; but to insure this hardiness the sheet aluminium of which the balance is made must be *very thin*. After several weeks' use the pumice may begin to look moist, and even slight traces of moisture may be seen on the outside of the lead cup, in consequence of watery vapour attracted by the sulphuric acid from the atmosphere; but the pumice should then be taken out and dried. At all events this must be done in good time, before enough of liquid has collected to give any tendency to drop. In all climates in which I have hitherto tested the instrument, I have found the pumice effective for insulation and safe in keeping all the liquid to itself for two months. But Mr. Becker having reported to me that many instruments have been returned to him in a ruinous condition from drops of sulphuric acid having become scattered through their metal work, I now cause to be engraved conspicuously on the outer case of the instrument "PUMICE DANGEROUS, IF NOT DRIED ONCE A MONTH;" also a frame carrying a card, on which the dates of drying are inscribed, to be placed in a convenient position on the roof of the instrument.

374. To prepare the instrument for use, the inner coating of

the Leyden jar must be charged through a charging rod, insulated in a vulcanite or glass tube, and let down for the occasion through a hole in the roof of the instrument, by aid of a small electrophorus, which generally accompanies the instrument, or by an electrical machine. I generally prefer to give a negative charge to the inner coating, as I have not found any physical reason, such as that mentioned in § 349 above, to prefer a positive charge to a negative charge; and the negative charge gives increased readings of the micrometer, in the ordinary use of the instrument, to correspond to positive charges of the principal electrode, as will be presently explained. Before commencing to charge the jar, the upper attracting-plate should be moved to nearly the highest position of its range by the micrometer screw, otherwise too strong a force of electric attraction may be put upon the aluminium square; and besides, the jar will discharge itself between the upper plate and the extreme edge of the aluminium square, when it is pulled very much above the level of the guard-plate by the electric attraction. I have not found any injury or change of electric value of the scale-divisions to arise from any such rough usage; but still, to guard against such a possibility, I propose to add to the guard-plate checks to prevent the corners of the aluminium from rising much, if at all, above its level, and to conduct the discharge and protect the aluminium and platinum from the shock, in case of the upper plate being brought too near the lower. When the instrument is being charged, or when it is out of use at any time, the umbrella should always be kept down; but it must be raised to insulate the principal electrode, of course, before proceeding to apply this to a body whose difference of potential from a body connected with the case of the instrument is to be measured.

375. In using the instrument the umbrella must very frequently be lowered, or metallic communication established in any other convenient way between the chief electrode and the outer brass case, the micrometer screw turned until the hair takes its sighted position, and the reading taken, the hundreds being read on the interior vertical scale, and the units (or single divisions of the circle) on the graduated circle above. The number thus found is called the earth-reading. It measures

the distance from an arbitrary zero position to the position in which the upper attracting-plate must be placed to give the amount of electric force on the aluminium square which balances the lever in its sighted position. A constant added to the earth-reading, or subtracted from it, gives (§ 341) a number simply proportional to the difference of potentials between the upper and lower plate; that is to say, between the two coatings of the Leyden jar. The vertical scale and micrometer circle are numbered, so that increased distances between the plates gives increased readings; and the zero reading should correspond as nearly as may be to zero distance between them; although in the instruments hitherto made no pains have been taken to secure this condition, even somewhat approximately. If it is desired to know the constant, an electrical experiment must be made to determine it, which is done with ease; but this is not necessary for the ordinary use of the instrument, which is as follows:—

376. First, an earth-reading is taken, then the upper electrode is insulated by raising the umbrella, or otherwise breaking connexion between the principal electrode and the outer metal case of the instrument. The principal electrode and the outer case are then connected with the two bodies whose difference of potential is to be determined, and the micrometer screw is turned until the hair is brought to its sighted position. The reading of hundreds on the vertical scale and units on the circle is then taken. Lastly, the principal electrode is again connected with the case of the instrument and another earth-reading is taken. If the second earth-reading differs from the first, the observer must estimate the most probable earth-reading for the moment when the hair was in its sighted position, with the upper plate and the metal case in connexion with the two bodies whose difference of potential is to be measured. The estimated earth-reading is to be subtracted from the reading taken in connexion with the bodies to be tested. This difference measures (§ 362) the required difference of potentials between them in units of the instrument. The value of the unit of the instrument ought to be known in absolute electrostatic measure; and the difference of reading found in any experiment is to be multiplied by this, which is called (§ 341) the

absolute coefficient of the instrument, to give the required difference of potentials in absolute measure. It so happens that, in the portable electrometers of the kind now described which have been hitherto constructed, the absolute coefficient is somewhere about ·01, so that one turn of the screw, or one hundred divisions of the circle, corresponds to somewhere about one electrostatic unit, with a gramme for the unit of mass, a centimetre for the unit of distance, and a second for the unit of time; but the different instruments differ from one another by as much as ten or twenty per cent. in their absolute coefficients. In all of these I have found between three and four Daniell's cells to correspond to the unit division; that is to say, between three hundred and four hundred cells to a full turn of the screw. With great care, the observer may measure small differences of potentials by this instrument to the tenth part of a division (or to about half a Daniell's cell). With a very moderate amount of practice and care, an error of as much as half a division may be avoided in each reading.

377. But there are imperfections in the instrument itself which make it difficult or impossible to secure very minute accuracy, especially in measurements through wide ranges.

(1.) In the first place, I am not sure that the end of the needle carrying the hair is protected sufficiently by the wire fences (§ 370) from electric disturbance to provide against any error from this source, which possibly introduces serious irregularities.

(2.) In the second place, the capacity of the jar in the small portable instrument is not sufficient to secure that the potential of its inner coating shall not differ sensibly with the different distances to which the upper plate is brought, to balance the aluminium lever with the hair in its sighted position. But on this point it is to be remarked that the electric density on the upper surface of the guard-plate is in its central parts always the same when the hair is in its sighted position; and it is therefore only the comparatively small difference of the quantity of electricity on this surface, towards the rim, corresponding to different distances of the attracted plate, that causes difference of potential in the inner coating of the jar. But if the upper attracting-plate be kept for several minutes at any distance, differing by a few turns of the screw, from that which brings the hair to its sighted position, the electricity creeps along the inner unconnected surface of the glass so as to diminish the charge of the inner metallic coating, or increase it, according as the distance is too great or too small. If then quickly the screw be turned and the earth-reading taken, it is found smaller or greater, as the case may be, than previously; but after a few minutes more it returns to its previous value very approximately. Error from this source may be practically avoided by taking care never to allow the hair to remain for more than a few minutes far from its sighted position; never so far, for instance, as above the centre of the upper, or below the centre of the lower spot.

(3.) A third source of error arises from change of temperature influencing the indications. In most of the instruments hitherto made I have found that the warmth of the hand produces in a few minutes a very notable augmentation of the earth-reading (as it were an increased charge in the jar); but in the last instrument which I have tested (White, No. 18) I find the reverse effect, the earth-reading becoming smaller as the instrument is warmed, or larger when it is cooled. I have ascertained that these changes are not due to changes in the electric capacities of the Leyden jars; and I have found that the change, if any, of specific inductive capacity of glass by change of temperature is excessively small, in comparison to what would be required to account for the temperature errors of these instruments, which probably must be due to thermo-elastic properties of the platinum wire, or of the stretching-springs, or of the aluminium balance-lever, or to a combination of the effects depending on such properties; but I have endeavoured in vain, for several years, and made many experiments, to discover the precise cause. It surely will be found, and means invented for remedying the error, now when I have an instrument in which the error is in the opposite direction to that of most of the other instruments. It is of course much greater in some instruments than in others: in some it is so great that the earth-reading is varied by as much as twenty divisions by the warmth of the hand in the course of five or ten minutes after commencing to use the instrument, if it has been previously for some time in a cold place. Its influence may be eliminated, not quite rigorously, but nearly enough so for most practical purposes, by frequently taking earth-readings (§ 375) and proceeding according to the directions of § 376.

(4.) A fourth fault in the portable electrometer is, that the diameter of the guard-plate and upper attracting disc, which ought to be infinite, are not sufficiently great, in proportion to the greatest distance between them, to render the scale quite uniform in its electric value throughout. A careful observer will, however, remedy the greater part of the error due to this defect, by measuring experimentally the relative (or absolute) values of the scale-division in different parts of the range. There will, however, remain uncorrected some irregularity, due to influence of the distribution of electricity over the uncoated inner surface, in the instruments as hitherto made, in all of which the inner surface of the jar is coated with tinfoil only below the guard-plate, so that the upper surface of the guard-plate may be seen clearly, in order that the observer may always see that all is in order about the aluminium square and aperture round it; and particularly that there are no injurious shreds or minute fibres. But the irregular influence of the electrification of the uncoated glass, if found sensible, will be rendered insensible by continuing the tinfoil coating an inch above the upper surface of the guard-plate.

378. All faults, except the temperature error, depend on the smallness of the instrument; and if the observer chooses to regard as portable an instrument of thirty centimetres (or a foot) diameter, with all other dimensions, and all details of construction, the same as those of the instrument described above, he may have a portable electrometer practically free from three of the four faults described. It is scarcely to be expected that a small instrument (12½ centimetres high, and 8½ centimetres in diameter) which may be carried about in the pocket can be free from such errors. But they are so far remedied as to be probably not perceptible, in the large stationary instrument which I now proceed to describe.

STANDARD ELECTROMETER.

379. This instrument (figs. 12, 13, and 14, Plate II.) differs from the portable electrometer only in dimensions, and in certain mechanical details, which are arranged to give greater accuracy by taking advantage of freedom from the exigencies of a small portable instrument. It is at present called the standard electrometer, in anticipation of either remedying, or of learning to perfectly allow for, the temperature error, and of finding by secular experiments on the elasticity of metals, that their properties used in the instrument are satisfactory as regards the permanence from year to year, and from century to century, of the electric value of its reading. It is an instrument capable of being applied with great ease to very accurate measurements of differences of potential, in terms of its own unit. The value of the unit for each such standard instrument ought, of course, to be determined with the greatest possible accuracy in absolute measure; and until confidence can be felt as to its secular constancy, determinations should frequently be made by aid of the absolute electrometer.

380. The Leyden jar of the standard electrometer consists of a large thin white-glass shade coated inside and outside to within 6 centimetres of its lip, and placed over the instrument as an ordinary glass shade, to protect against dust, currents of air, and change of atmosphere. It may be removed at pleasure from the cast-iron sole of the instrument, and then the interior works are seen, consisting of—

(1.) A continuous disc of brass supported on a glass stem, in prolongation of a stout brass rod or tube sliding vertically in Vs, in which it is kept by a spring [better by two springs (§ 369)], and resting with its lower flat end on the upper end of a micrometer screw shaft, shown in fig. 13, where the screw, graduated circle, and stout brass rod are as seen in the instrument; the manner in which the lower end of the rod or tube is constructed to keep the round upper end of the screw-shaft in position is shown in section in fig. 14.

(2.) Resting on three glass columns, a guard-plate with a square aperture in its centre, and carrying on its upper side the stretching springs and thin platinum wire suspension of an aluminium balance-lever, shaped like those of the gauge (§ 353) and the portable (§ 368) already described, but somewhat larger. The tops of the three glass columns are rounded; a round hole and a short slot in line with this hole are cut in the guard-plate, and receive the rounded ends of two of the columns, which are somewhat longer than the third. The flat smooth lower surface of the guard-plate rests simply on the top of the third glass column. The diameter of the round hole and the breadth of the slot in the guard-plate may be about $\frac{1}{\sqrt{2}}$ of the diameter of curvature of the upper hemispherical rounded ends of the glass columns, so that the bearing portions of the rounded ends in the round hole and in the slot respectively may be inclined somewhere about 45° to the plane of the plate. This well-known but too often neglected geometrical arrangement gives perfect steadiness to the supported plate, without putting any transverse strain upon the supporting glass columns, such as was almost inevitable, and caused the breakage of many glass stems, before the mental inertia opposing deviations from the ordinary instrument-maker's plan (of screwing the guard-plate to brass mountings cemented to the tops of the glass columns) was overcome. It has also the advantage of allowing the guard-plate to be lifted off and replaced in a moment.

(3.) Principal electrode projecting downwards through a hole in the sole of the instrument, and rigidly supported from above by a brass mounting cemented to the top of a thick vertical glass column, connected by a light spiral spring with the lower attracting plate moved up and down by the micrometer screw. The aperture round the principal electrode may be ordinarily stopped by a perforated column of well paraffined vulcanite projecting some distance above and below the aperture, which I find to insulate extremely well, even in the smoky, dusty, and acidulated atmosphere of Glasgow. When an extremely perfect insulation of the principal electrode and connected attracting plate is required, the vulcanite stopper surrounding it may be withdrawn from the aperture, so that the only communication between the electrode and the case of the instrument may be along the two glass columns in the artificially dried interior atmosphere of the case; but from day to day, when the instrument is out of use, the aperture round the principal electrode should be kept carefully stopped, if not by a vulcanite insulator by a perforated cork; (although I find but little loss of insulation, either along the inner glass surface of the Leyden jar or along the three glass columns, when this precaution is neglected).

(4.) Temporary charging-rod enclosed in and supported by a vertical insulating column of paraffined vulcanite, or a glass tube well varnished outside and thickly paraffined inside. This insulating column bearing the charging-rod is turned round till a horizontal spring projecting from its upper end touches the inner coating of the jar, when this is to be charged from an independent source, or when, for any other experimental reason, it is to be put in connexion with a conductor outside the case of the instrument.

(5.) A small replenisher of the kind described for the quadrant electrometer (§ 352), but with much wider air-spaces to prevent discharge by sparks.

(6.) A large glass or lead dish to hold as large masses of pumice as may be, which are to be kept sufficiently impregnated with strong sulphuric acid.

381. A considerable portion of the jar above the guard-plate is left uncoated to allow the observer to see easily the hair and white background with black dots; also several other smaller parts of the glass above the guard-plate are left uncoated to admit light to allow a small circular level on the upper side of the guard-plate to be seen. The long arm of the aluminium balance-lever is very thoroughly guarded by double cages and fences of wire (§ 370), so that it can experience no sensible influence from electric disturbing forces when the covering jar is put in position and electric connexion is established between its inner coating and the guard-plate by projecting flexible wires or slips of metal.

382. The aluminium square plate is somewhat larger, and the platinum bearing wire somewhat longer in this instrument than in the portable electrometer, to render it sensible to smaller differences of potential. The step of the screw is the same as in the portable ($\frac{1}{50}$ of an inch), and one division ($\frac{1}{100}$ of the

circumference of the screw-head) corresponds to a difference of potentials which, roughly speaking, is equal to about that of a single cell of Daniell's. The effective range of the instrument is about sixty turns of the screw, and therefore about 6000 cells of Daniell's. That of the portable electrometer is about 15 turns of the screw (equivalent to about 5000 cells). Neither of these instruments has sufficient range to measure the potential to which Leyden jars are charged in ordinary electric experiments, or those reached by the prime conductor of a powerful electric machine. The stationary instrument with its long screw and its large plates now described, would go far towards meeting this want if its aluminium lever and platinum suspension were made on the same scale as those of the portable electrometer; but for an instrument never wanted to directly measure differences of potentials of less than two or three thousand cells, the heterostatic (§ 385) principle is in general not useful, and therefore I have constructed the following very simple idiostatic (§ 385) instrument, which is adapted to measure with considerable accuracy differences of potential from 4000 cells upwards, to about 80,000 cells.

LONG-RANGE ELECTROMETER.

383. In this (fig. 15, Plate VI.) the continuous attracting-plate is above, and the guard-plate with aluminium balance below, as in the portable electrometer; but, as in the standard stationary electrometer, the upper plate is fixed and the lower plate is moved up and down by a micrometer-screw. The mechanism of the screw and slide has all the simplicity and consequent accuracy of that of the standard electrometer. In the only long-range instrument yet constructed the step of the screw is the same as that of the others ($\frac{1}{50}$ of an inch). In future instruments it would be well either to have a longer step or to have a simple mechanism (which can be easily added) to give a quick motion; as in the use of the present instrument, the turning of the screw required for great changes of the potential measured is very tedious. The guard-plate projects by more than an inch all round beyond the rim of the upper attracting-plate; partly to obviate the necessity of giving it a

thick rim, which would be required to prevent brushes and sparks from originating in it, if it had only the same diameter as the continuous plate above, and partly to guard the observer from receiving a spark or shock in measuring the potential of an electric machine or of a Leyden battery, and to prevent his hair from being attracted to the upper plate. Thus the guard-plate is allowed to be no thicker than suffices for stiffness, and this allows the observer to see the hair at the end of the aluminium balance-lever without the lever being made of a dynamically disadvantageous shape, as would be necessary if the guard-plate were thick, or had a thick rim added to it. No glass case is required for this instrument. The smallness of the needle and the greatness of the electric force acting on it are such that I find in practice no disturbance to any inconvenient degree by ordinary currents of air; although it and all these attracted disc instruments show the influence of sudden change of barometric pressure, such as that produced by opening or shutting a door. If not kept under a glass shade when out of use, the lower surface of the upper attracting-plate, and the lower surface of the guard-plate and attracted aluminium square, should be carefully dusted by a dry cool hand. Generally speaking, none of the vital electric organs of an electrometer should be touched by a cloth, as this is almost sure to leave shreds fatal to their healthy action.

[(Addition, 1870) I intend to cover the whole instrument with a glass shade, well varnished over a large space round an aperture in its top, into which an insulated electrode for the upper plate will be cemented: because with the instrument open as it is at present great difficulty has been experienced in measuring high tension on account of dust and shreds which impair the insulation.]

384. The effective range of this instrument is about 200 turns of the screw. Rather greater force of torsion is given than in the portable electrometer, and a rather smaller attracted disc may be used, so that upwards of four cells may be the electric value of one division. The instrument in its present state measures nearly but not quite the highest potential I can ordinarily produce in the conductor of a good Winter's electric machine, which sometimes gives sparks and brushes a foot long.

385. The classification of electrometers given above is founded on the shape and kinematic relations of their chief organic parts; but it will be remarked that another principle of classification is presented by the different electric systems used in them, which may be divided into two classes:—

I. Idiostatic, that in which the whole electric force depends on the electrification which is itself the subject of the test.

II. Heterostatic, in which, besides the electrification to be tested, another electrification maintained independently of it is taken advantage of.

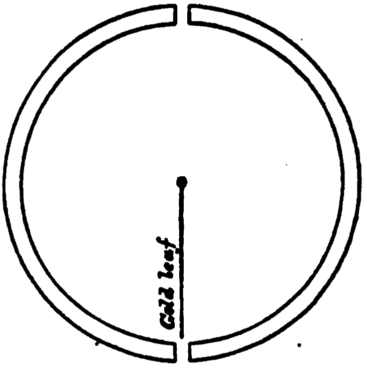
Thus, for example, the long-range electrometer (§§ 383, 384) is simply idiostatic, and is not adapted for heterostatic use; but each of them may be used idiostatically. The absolute electrometer was at first simply idiostatic (§§ 358-362); more recently it has been used heterostatically, and is about to acquire (§ 363) special organs adapted for heterostatic use; as yet, however, no species of the absolute electrometer promising permanence has come into existence. [See §§ 364-367 describing a heterostatic absolute electrometer of a species which (Jan. 1871) promises to be permanent.]

386. It is instructive to trace the origin of various heterostatic species of electrometers by natural selection. A body hanging, or otherwise symmetrically balanced, in the middle of a symmetrical field of force, but free to move in one direction or the other in a line tangential to a line of force, moves in one direction or the opposite when electrified positively or negatively. Bohnenberger's arrangement of this kind has a convenient and approximately constant field of force; and his instrument was chosen in preference to others which may have been equally sensitive, but were less convenient and constant, and it became a permanent species.

387. Bennet's gold-leaf electroscope, constructed with care to secure good insulation, electrified sufficiently to produce a moderate divergence, has been often used to test, by aid of this electrification, the quality of the electrification of an electrified body brought into the neighbourhood of its upper projecting electrode, causing, if its electricity is of the same sign as that of the gold leaves, increase of divergence; if of the opposite sign, diminution. By connecting the upper electrode with the

inner coating of a Leyden jar with internal artificially dried atmosphere, the charge of the gold leaves may be made to last with little loss from day to day; and by insulating Faraday's metal cage (§ 342) round the gold leaves, and alternately connecting it with the earth and with a conductor whose difference of potentials from the earth is to be tested, an increase or a diminution of divergence is observed according as this difference is negative or positive, the gold leaves being positive. Hence (through Peltier's and Delmann's forms) the heterostatic stationary and portable repulsion electrometers, described (§§ 274-277, 263 above) in the Royal Institution Lecture on "Atmospheric Electricity," and in Nichol's *Cyclopædia*, article "Electricity, Atmospheric," already referred to, of which one species still survives in King's College, Nova Scotia, and in the Natural Philosophy Classroom of Edinburgh University. The same form of the heterostatic principle applied to Snow Harris's attracted disc electrometer gave the portable and standard electrometers described above.

388. A modification of Bohnenberger's electroscope, in which the two knobs on the two sides of the hanging gold leaf became transformed into halves of a circular cylinder, with its axis horizontal and the gold leaf hung on a wire insulated in a position coinciding with its axis; producing a species designed for telegraphic purposes, but which did not acquire permanence by natural selection, and is only known to exist in one fossil specimen. In this instrument the wire bearing the gold leaf was connected with a charged Leyden jar, and the semi-cylinders with the bodies whose difference of potential was to be tested. But various modifications of the divided-cylinder or divided-ring class with the axis vertical and plane of motion horizontal have done some practical work, and one species, the new quadrant electrometer (§ 346), promises to become permanent.



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